Chapter 13: Rotation of a Rigid Body

The rigid body model:

Practitioners of other sciences often poke fun at physicists who stereotypically start off a class by asking you to "Consider a spherical cow…"

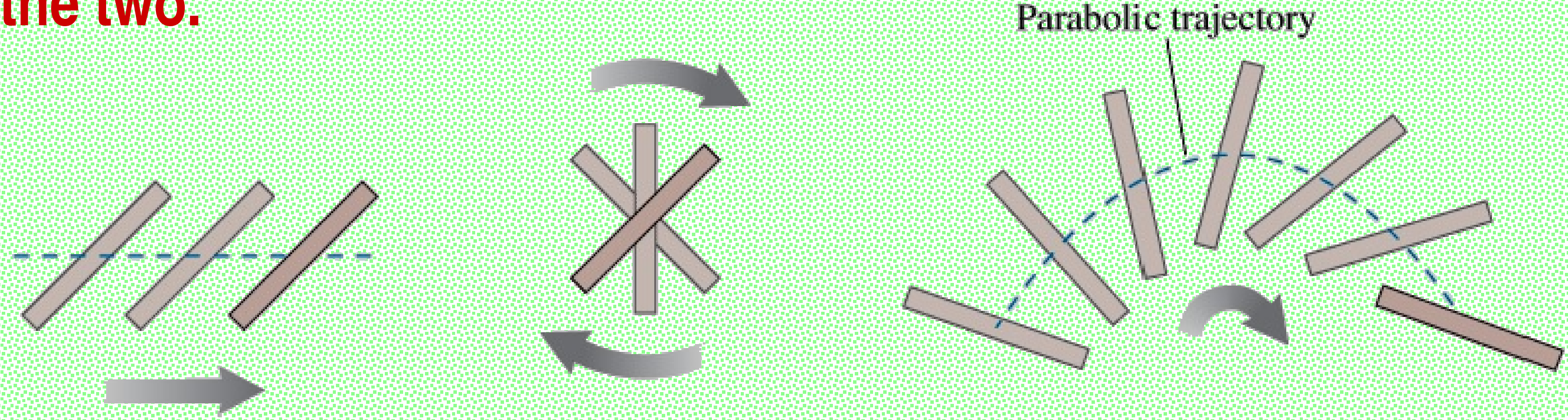
In fact, this is the "particle model", which has actually served us quite well…until now.

In the particle model, the structure (distribution of matter) makes no difference to the analysis. **However, for rotating objects, the distribution of matter is key.**

This chapter introduces the "rigid body model", in which all parts of an object rotate with the same angular velocity (a.k.a. angular frequency), ω.

Chapter 13: Rotation of a Rigid Body

In rigid body dynamics we have two types of motion: translational and rotational, plus a third which is a combination of the two.



Translational motion: The object as a whole moves along a trajectory but does not rotate.

Rotational motion: The object rotates about a fixed point. Every point on the object moves in a circle.

Combination motion: An object rotates as it moves along a trajectory.

So far, we have only considered translational motion. This chapter shows us how to include rotation into the dynamics.

Chapter 13: Rotation of a Rigid Body

Rotational kinematics; a reminder:

In Chapter 7, we introduced the rotational analogues of displacement (***x*:*θ***), velocity (***v*:*ω***), and acceleration (***a*:*α***)

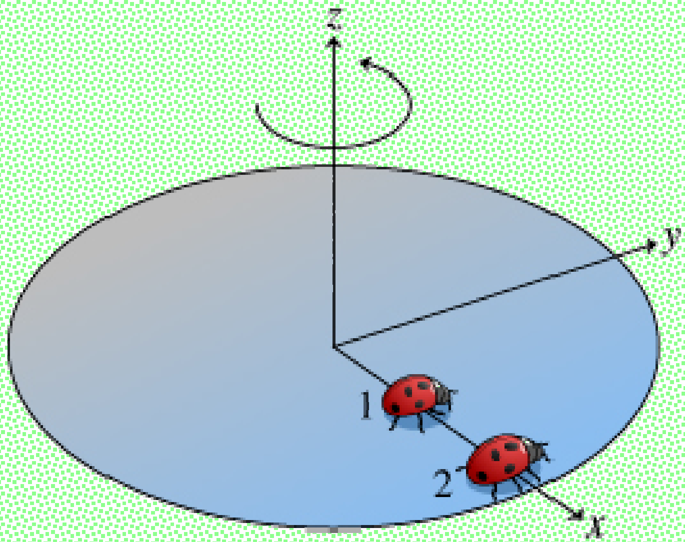
TABLE 13.1 Rotational and linear kinematics for constant acceleration

Rotational kinematics	Linear kinematics
$\omega_f = \omega_i + \alpha\Delta t$	$v_f = v_i + a\Delta t$
$\theta_f = \theta_i + \omega_i\Delta t + \frac{1}{2}\alpha(\Delta t)^2$	$x_f = x_i + v_i\Delta t + \frac{1}{2}a(\Delta t)^2$
$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$	$v_f^2 = v_i^2 + 2a\Delta x$

$v = \omega r$, $a_r = \omega^2 r$, and $a_t = \alpha r$, where r is the instantaneous radius of curvature (= radius of circle for circular motion).

Clicker question 13.1

Two ladybugs sit on a rotating disc without slipping. Ladybug 1 is half way between the rotation axis and ladybug 2. The **angular speed, *ω*,** of ladybug 1 is:



a) half that of ladybug 2;

b) the same as ladybug 2;

c) twice that of ladybug 2;

d) impossible to determine from the information given.

Chapter 13: Rotation of a Rigid Body

Clicker question 13.1

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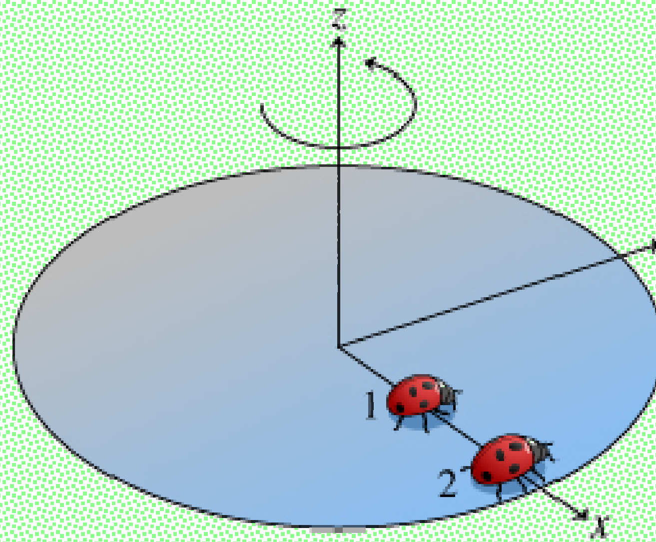


a) half that of ladybug 2;

b) the same as ladybug 2; ✔

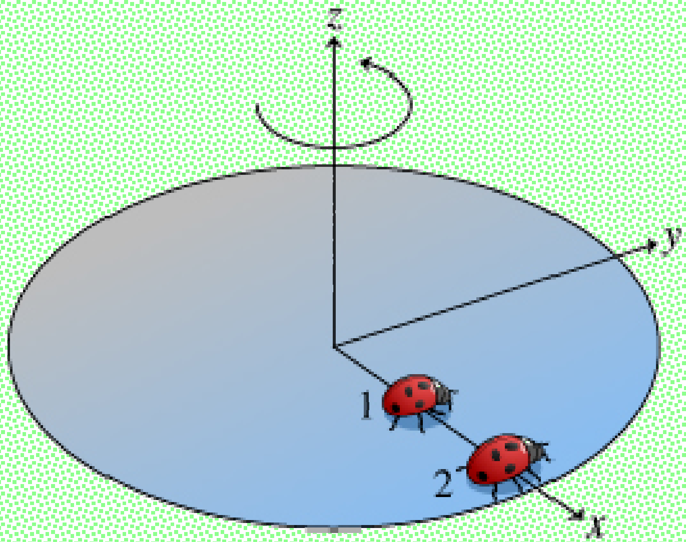
c) twice that of ladybug 2;

d) impossible to determine from the in… en.

Rigid body rotation means the angular speed, ω, is constant.

Clicker question 13.2

Two ladybugs sit on a rotating disc without slipping. Ladybug 1 is half way between the rotation axis and ladybug 2. The **linear speed, *v*,** of ladybug 1 is:



a) half that of ladybug 2;

b) the same as ladybug 2;

c) twice that of ladybug 2;

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Chapter 13: Rotation of a Rigid Body

Clicker question 13.2

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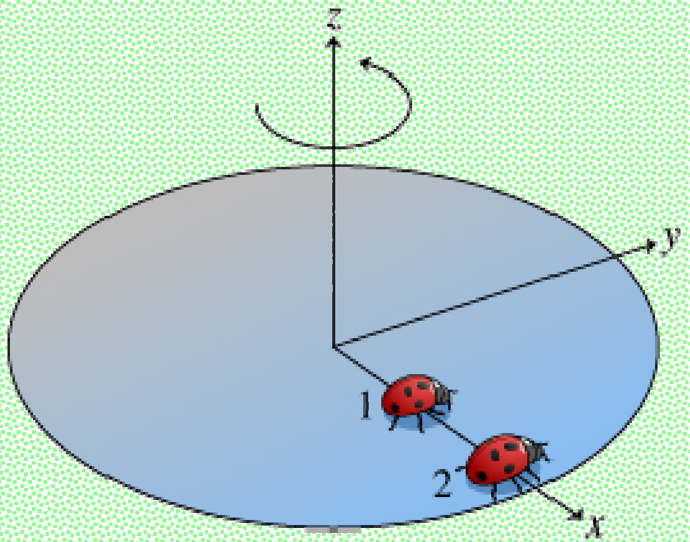


a) half that of ladybug 2; ✔

b) the same as ladybug 2;

c) twice that of ladybug 2;

d) impossible to determine from the information given.

Chapter 13: Rotation of a Rigid Body

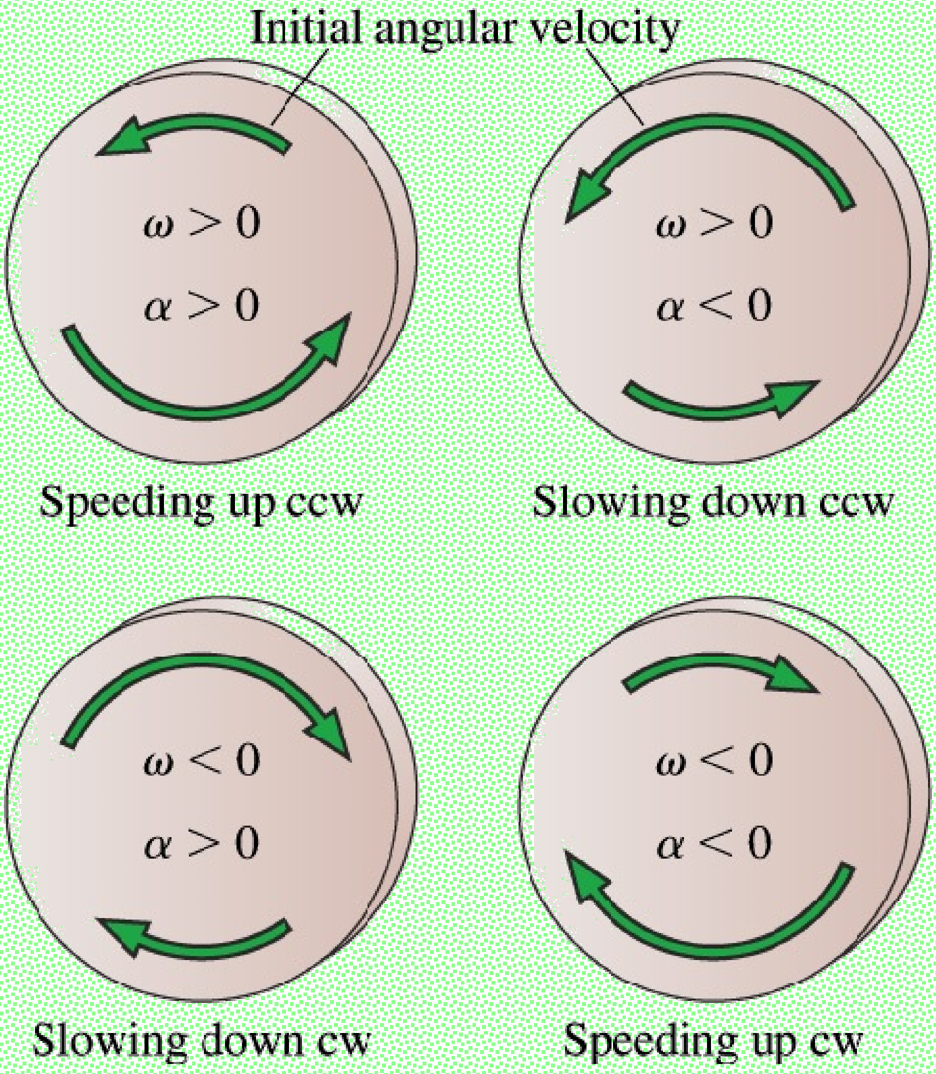
Sign convention for rotational kinematical quantities:

counterclockwise (ccw): +

clockwise (cw): –

α has same sign as ω if ω is increasing

α has opposite sign as ω if ω is decreasing.



Example: a review problem

A small dot is painted on the edge of a magnetic computer disk with radius 4.0 cm. Starting from rest, the disk accelerates at 600 rad s^{-2} for 0.5 s, then coasts at a steady angular velocity for another 0.5 s.

a) What is the speed of the dot at t = 1.0 s?

$\omega = \omega_0 + \alpha t$ until t = 0.5, then ω stays constant.

$\Rightarrow \quad \omega = 0 + 600\,(0.5) = 300$ rad s^{-1}

$v = \omega r = (300)(0.040) = \underline{12\ ms^{-1}}$

b) Through how many revolutions does the dot turn?

For the first 0.5s: $\Delta\theta_1 = \omega_0 t + \frac{1}{2}\alpha t^2 = (600)(0.5)^2/2 = 75$ rad

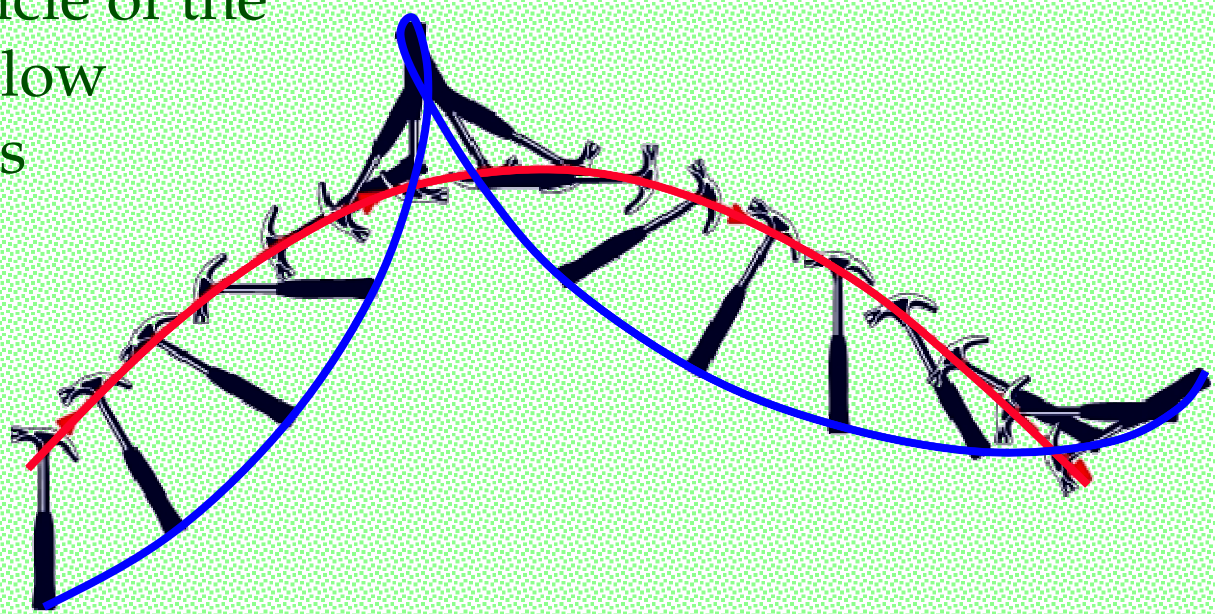
For the next 0.5s: $\Delta\theta_2 = \omega t = (300)(0.5) = 150$ rad

Total angular displacement: $\Delta\theta = \Delta\theta_1 + \Delta\theta_2 = 225$ rad = $\underline{35.6 \text{ revolutions}}$

Chapter 13: Rotation of a Rigid Body

Definition (HRW*): The ***centre of mass*** of an object or of a system of objects is that point which moves as though all mass were concentrated there and all forces were applied there.

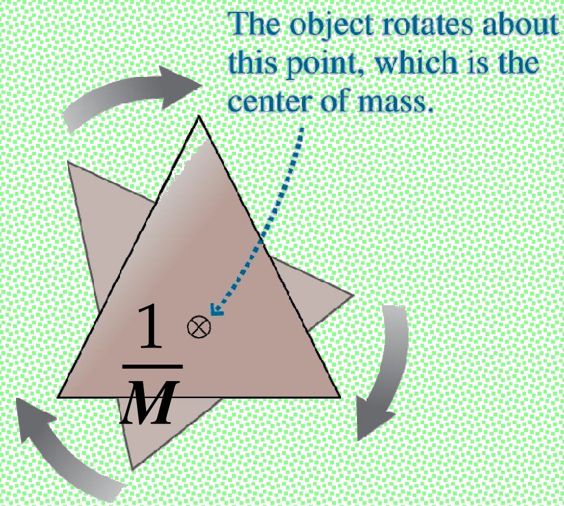
e.g., As a hammer tossed through the air spins handle over head, only the centre of mass follows the parabolic trajectory (red path) that a particle of the same mass would follow under the same forces (in this case gravity). The trajectory of the handle (blue path) is rather more complicated.



*Halliday, Resnick, and Walker: *Fundamentals of Physics*

Chapter 13: Rotation of a Rigid Body

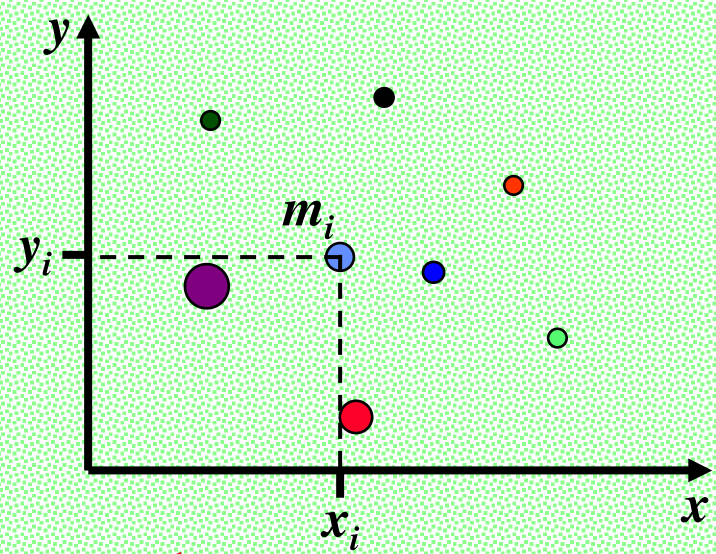
Knight's (less precise) definition: An unconstrained object (***i.e.,*** one not on an axle or a pivot) on which there is no net force rotates about a point called the centre of mass.



Locating the centre of mass for discrete particles…

$$x_{cm} = \frac{1}{M}\sum_i m_i x_i = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

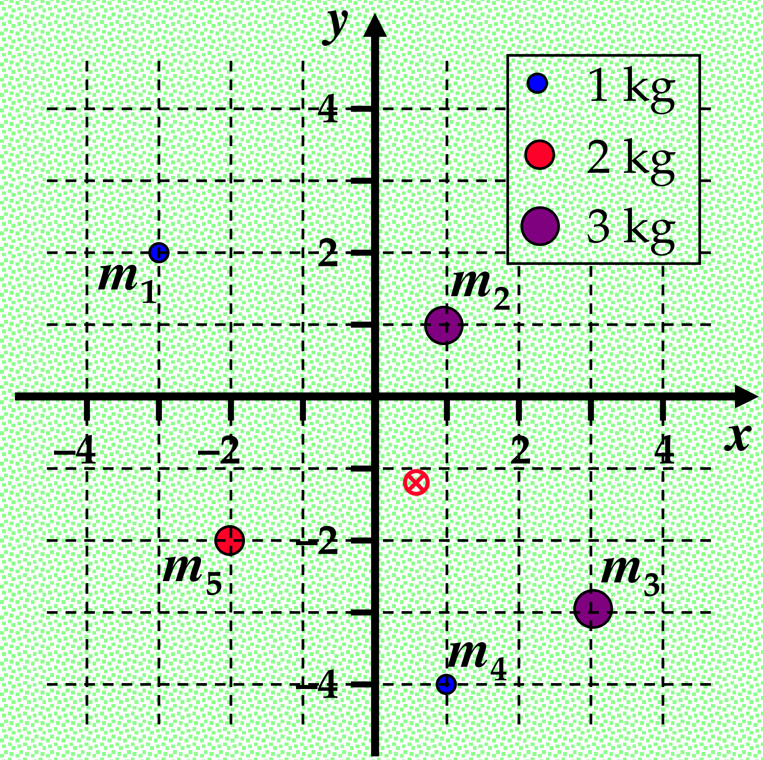
$$y_{cm} = \frac{1}{M}\sum_i m_i y_i = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$



and a similar expression for a *z*-component.

Chapter 13: Rotation of a Rigid Body

Example: Find the centre of mass for the system of the 5 objects shown.



The centre of mass is the position vector: $\vec{r}_{cm} = (x_{cm}, y_{cm})$, where

$$x_{cm} = \frac{1}{M}\sum_{i=1}^{N} x_i m_i; \quad y_{cm} = \frac{1}{M}\sum_{i=1}^{N} y_i m_i$$

$$M = m_1 + m_2 + m_3 + m_4 + m_5 = 10 \text{ kg}$$

$$\sum_{i=1}^{5} x_i m_i = (-3)(1) + (1)(3) + (3)(3) + (1)(1) + (-2)(2) = 6$$

$$\sum_{i=1}^{5} y_i m_i = (2)(1) + (1)(3) + (-3)(3) + (-4)(1) + (-2)(2) = -12$$

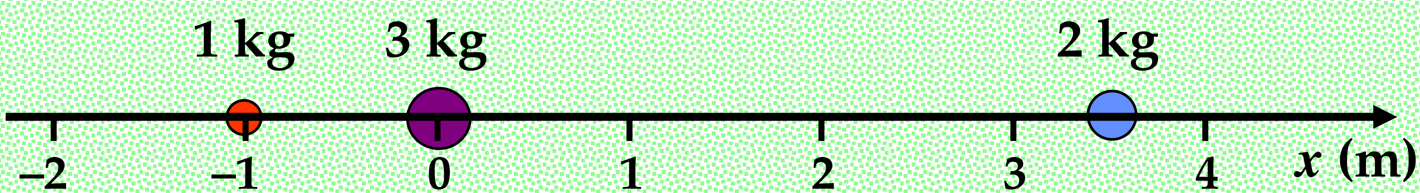
$$\Rightarrow \quad r_{cm} = (0.6, -1.2)$$

Clicker question 13.3

Where is the centre of mass for the system of three masses shown?

centre of mass formula: $x_{cm} = \frac{1}{M}\sum_{i=1}^{N} x_i m_i$

a) $x_{cm} = -1$ m	b) $x_{cm} = 0$ m	c) $x_{cm} = 1$ m
d) $x_{cm} = 2$ m	e) $x_{cm} = 3$ m	f) $x_{cm} = 3.5$ m



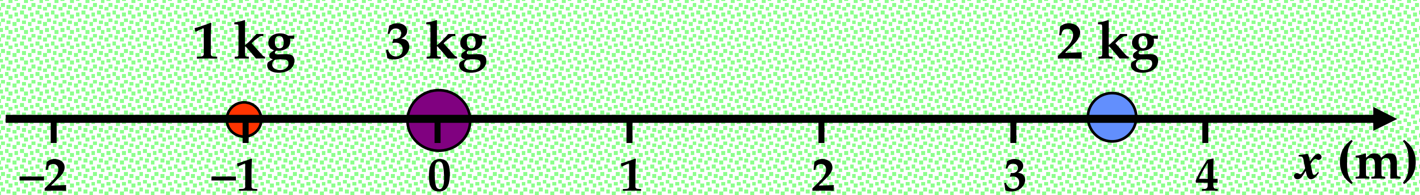
Chapter 13: Rotation of a Rigid Body

Clicker question 13.3

Where is the centre of mass for the system of three masses shown?

$$x_{cm} = \frac{1}{M}\sum_{i=1}^{3} x_i m_i = \frac{1}{6}\big((-1)(1) + (0)(3) + (3.5)(2)\big) = 1\text{ m}$$

a) $x_{cm} = -1$ m	b) $x_{cm} = 0$ m	c) $x_{cm} = 1$ m ✔
d) $x_{cm} = 2$ m	e) $x_{cm} = 3$ m	f) $x_{cm} = 3.5$ m

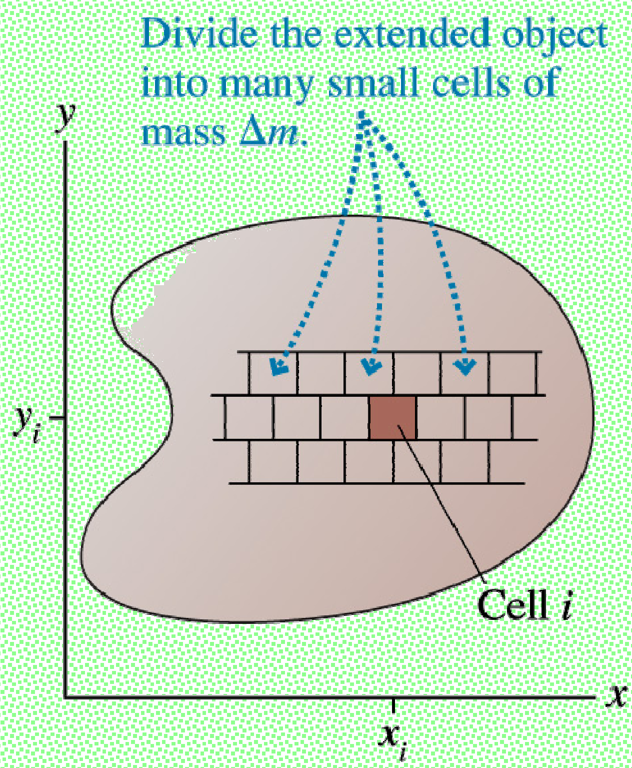


Chapter 13: Rotation of a Rigid Body

Locating the centre of mass for an "extended object"…

Imagine the extended object of mass ***M*** broken up into ***N*** smaller pieces each of mass **Δ*m***, and apply the sum formulae of the previous slide:

$$x_{\text{cm}} = \frac{1}{M}\sum_{i=1}^{N} x_i \,\Delta m; \quad y_{\text{cm}} = \frac{1}{M}\sum_{i=1}^{N} y_i \,\Delta m$$



Then, take $N \to \infty$, and the sums become integrals:

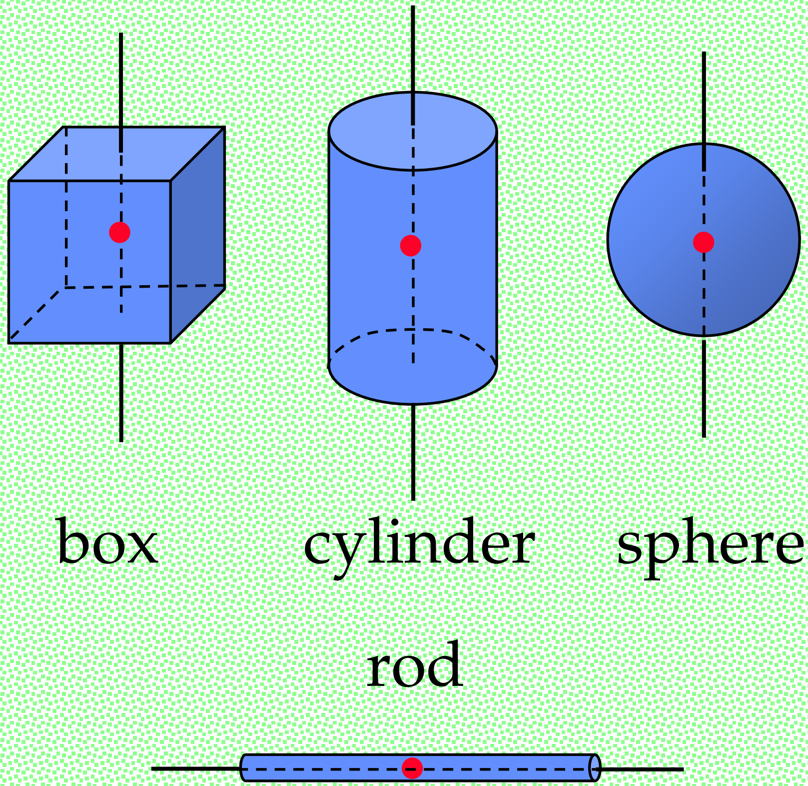
$$x_{\text{cm}} = \frac{1}{M}\int x\,dm \quad \text{and} \quad y_{\text{cm}} = \frac{1}{M}\int y\,dm$$

To evaluate these integrals, one must know how the mass is distributed in space, ***i.e.***, $m(x,y,z)$.

Chapter 13: Rotation of a Rigid Body

However, for our purposes, we *almost* never have to do these integrals!

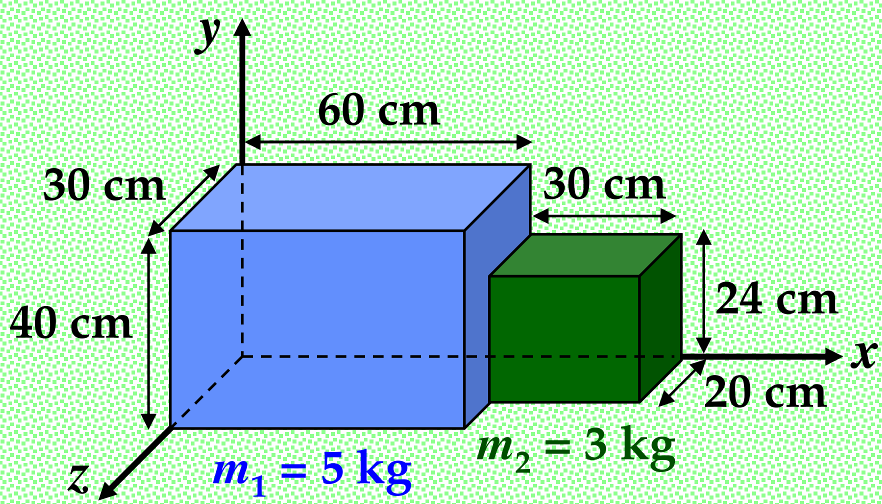
For uniform symmetric objects (***e.g.,*** sphere, cylinder, cube, rod, ***etc***.), the centre of mass is at the object's ***geometric centre*.**



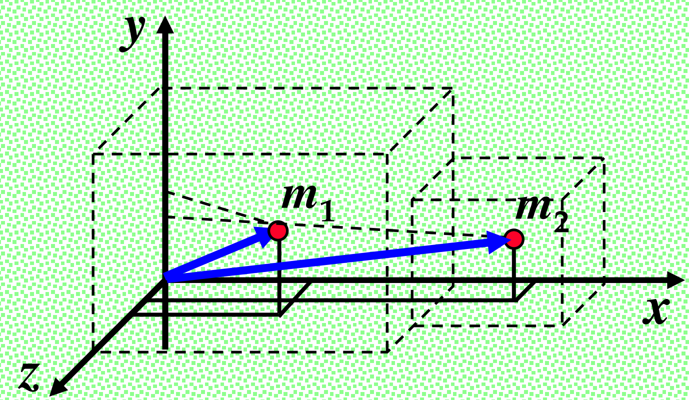
Centre of gravity: If you could balance an object by its ***centre of gravity***, it would remain in place without any other means of support. ***For objects with uniform density, the centres of mass and gravity are the same point.*** For a non-uniform object, these two points are, in fact, different.

Chapter 13: Rotation of a Rigid Body

example: Find the centre of mass of the system below consisting of two uniform boxes.



Strategy: Replace each symmetric object with a point mass at its centre of mass.



Using box dimensions as *x*,*y*,*z* coordinates:

$\vec{r}_{cm,1} = (30, 20, 15)$; $\vec{r}_{cm,2} = (75, 12, 10)$

$$\Rightarrow \vec{r}_{cm} = \frac{m_1\vec{r}_{cm,1} + m_2\vec{r}_{cm,2}}{m_1 + m_2} = \frac{1}{8}\big(5(30, 20, 15) + 3(75, 12, 10)\big) = (46.9, 17, 13.1)$$

Chapter 13: Rotation of a Rigid Body

13.3 Torques In addition to ***translational acceleration***, a force can cause ***angular acceleration***. The ability of a force to cause something to rotate is called a **torque (τ)**.

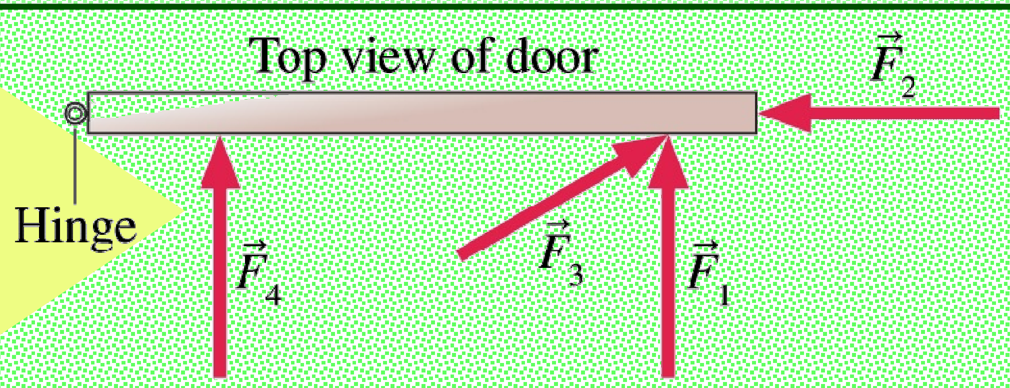
Torque is a vector quantity that depends upon:

1. the magnitude of the applied force, $\vec{F}$

2. The distance, r, connecting the point about which the object rotates (the "pivot point") and where $\vec{F}$ is applied, and

3. The angle between $\vec{r}$ and $\vec{F}$.

The ability of a force to open a door (and thus to rotate) depends not only on the magnitude of the force, but also where and in what direction the force is applied.

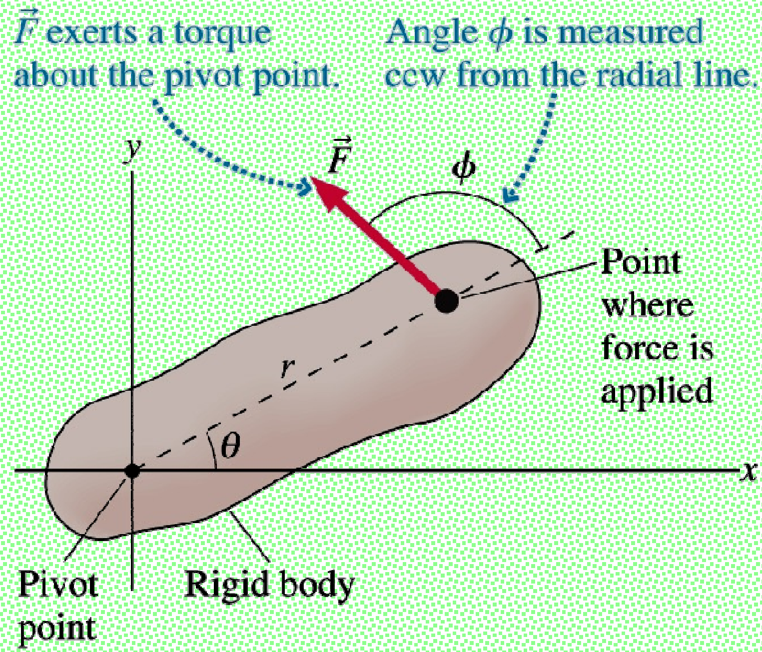


Chapter 13: Rotation of a Rigid Body

Provisional mathematical definition of torque:

$$\tau = rF\sin\phi$$

Units of τ: Nm. Formally, this is a Joule (J). However, since torque has nothing to do with energy, we always use Nm as the units for torque, never J.



Why provisional? **We'll "upgrade" to the "proper" definition of a torque (involving "cross products") by the end of the chapter.**

Sign convention:

$\tau > 0$ when F tends to rotate object counter-clockwise (ccw) about pivot.

$\tau < 0$ when F tends to rotate object clockwise (cw) about pivot.

Note: Torques depend very much on the location of the pivot!

Chapter 13: Rotation of a Rigid Body

Clicker question 13.4

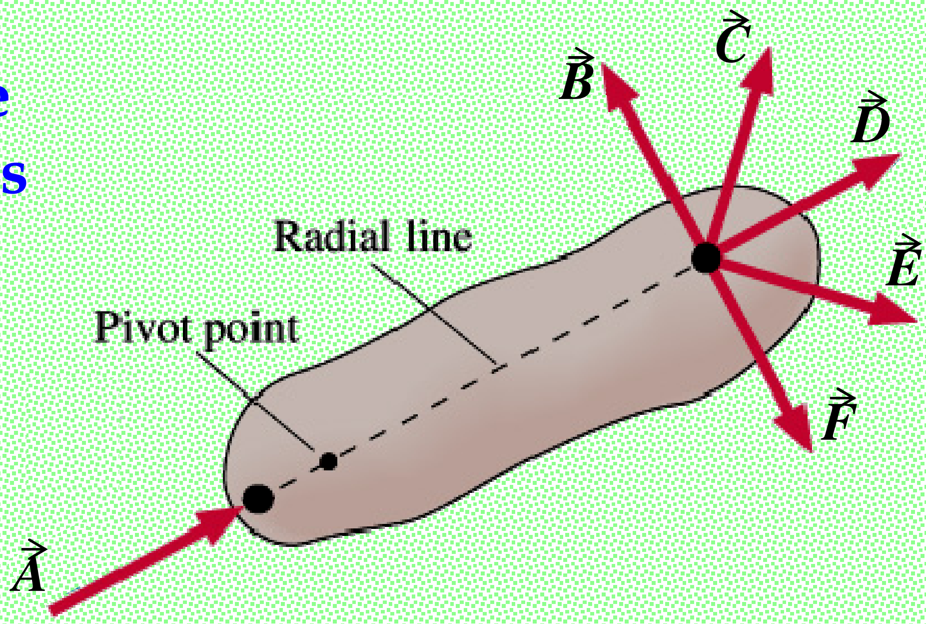
Six forces, $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$, $\vec{E}$, and $\vec{F}$, each with equal magnitude, are applied to a rigid body confined to rotate about the pivot point shown. Which forces produce the torques with the *greatest magnitude*?

a) All torques have the same magnitude because all forces have the same magnitude.

b) $\vec{A}$ and $\vec{D}$

c) $\vec{C}$ and $\vec{E}$

d) $\vec{B}$ and $\vec{F}$



Clicker question 13.4

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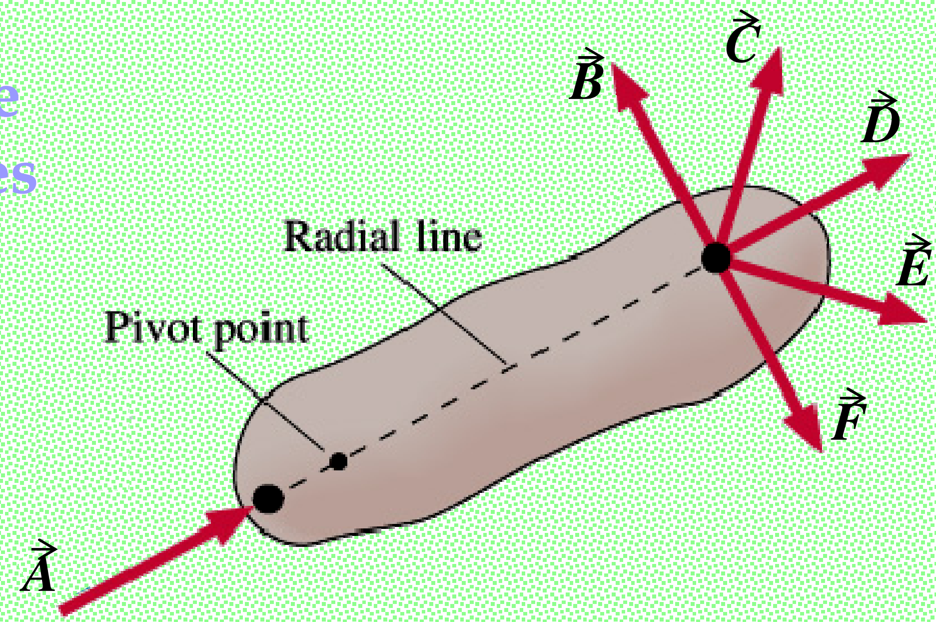


Clicker question 13.5

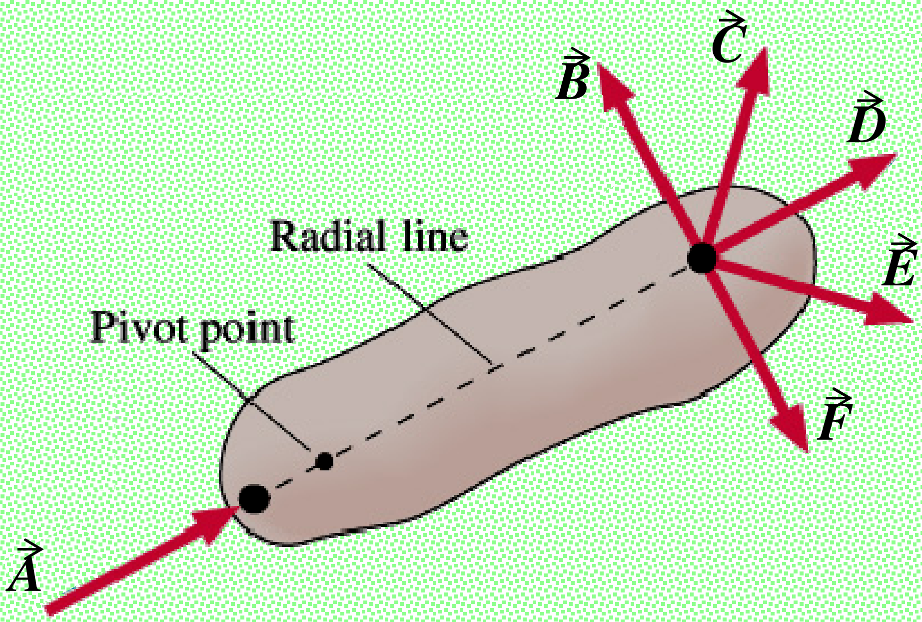
Six forces, $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$, $\vec{E}$, and $\vec{F}$, each with equal magnitude, are applied to a rigid body confined to rotate about the pivot point shown. Which forces produce no torque?

a) All forces produce torque since all forces have a non-zero magnitude.

b) $\vec{A}$ and $\vec{D}$

c) $\vec{C}$ and $\vec{E}$

d) $\vec{B}$ and $\vec{F}$



Clicker question 13.5

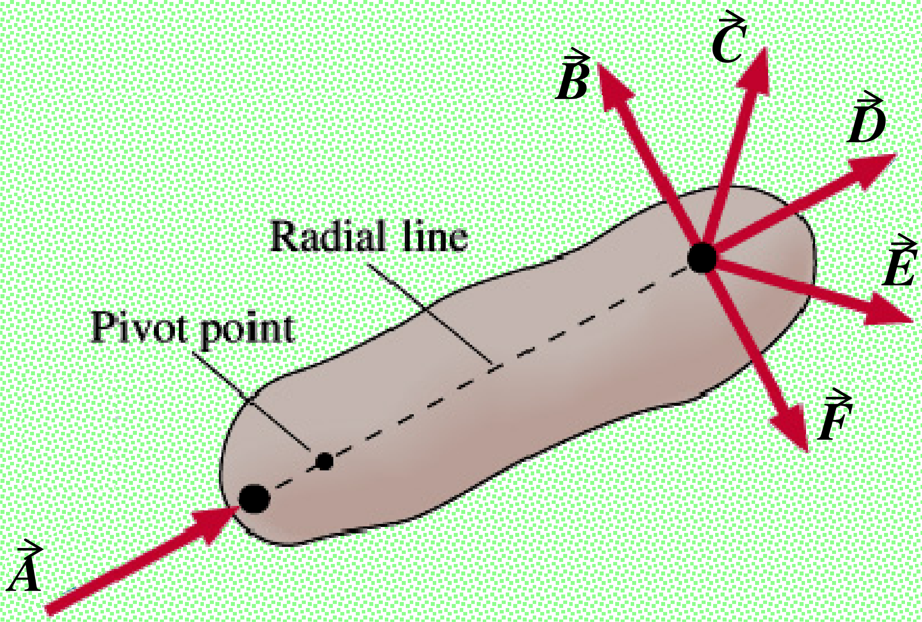
Six forces, $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$, $\vec{E}$, and $\vec{F}$, each with equal magnitude, are applied to a rigid body confined to rotate about the pivot point shown. Which forces produce no torque?

a) All forces produce torque since all forces have a non-zero magnitude.

b) $\vec{A}$ and $\vec{D}$ ✔

c) $\vec{C}$ and $\vec{E}$

d) $\vec{B}$ and $\vec{F}$



Clicker question 13.6

Six forces, $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$, $\vec{E}$, and $\vec{F}$, each with equal magnitude, are applied to a rigid body confined to rotate about the pivot point shown. For which forces is $\tau > 0$?

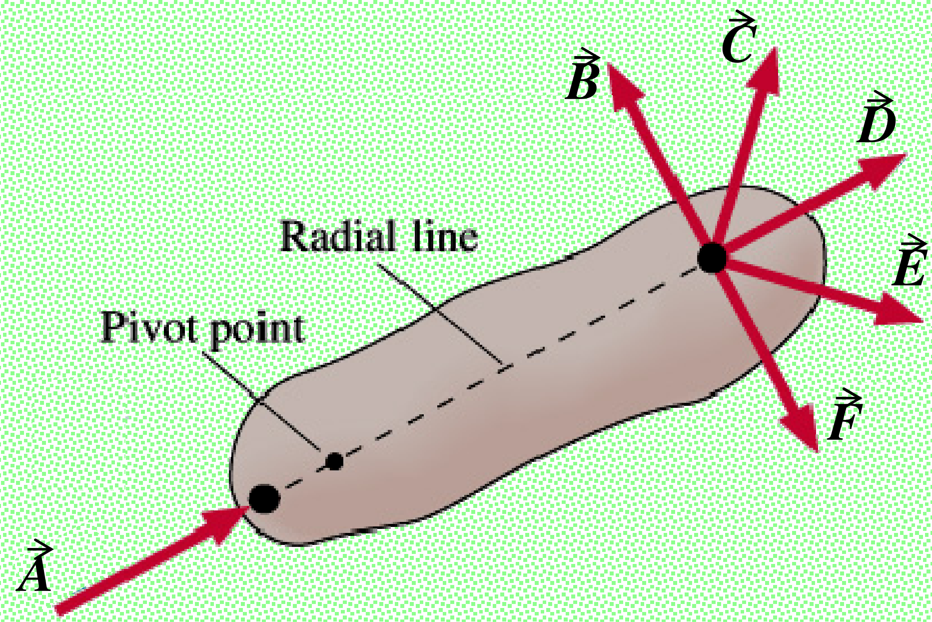
a) $\vec{B}$, $\vec{C}$, $\vec{E}$, and $\vec{F}$.

b) $\vec{B}$ and $\vec{C}$

c) $\vec{E}$ and $\vec{F}$

d) $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$

e) $\vec{B}$ only



Chapter 13: Rotation of a Rigid Body

Clicker question 13.6

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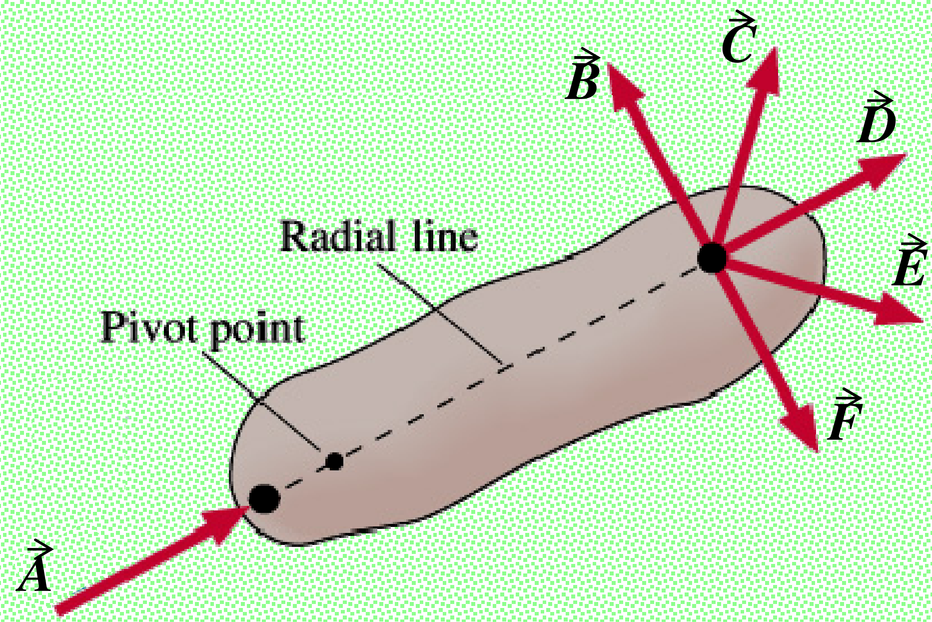
b) $\vec{B}$ and $\vec{C}$ ✔

c) $\vec{E}$ and $\vec{F}$

d) $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$

e) $\vec{B}$ only

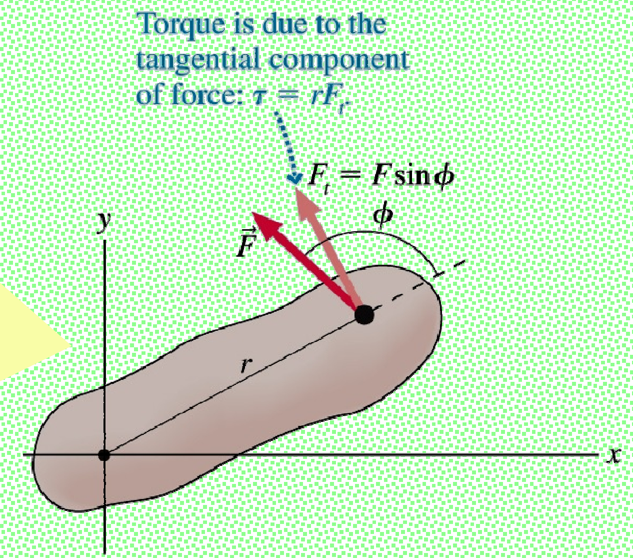


Chapter 13: Rotation of a Rigid Body

Two ways to think about torque…

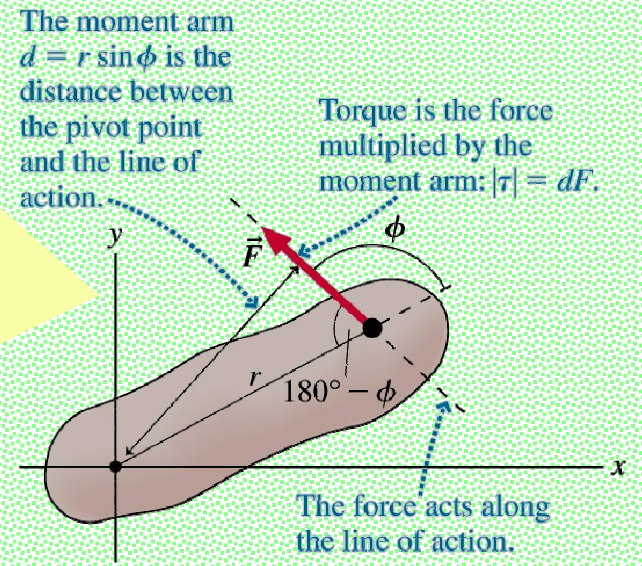
1. $\tau = r(F\sin\phi) = rF_t$

F_t is the *tangential component* of the force. Only the tangential component is responsible for torque; the radial component does not cause rotation.



2. $\tau = F(r\sin\phi) = Fd$

$d = r\sin\phi$ is the *moment arm* (lever arm). d is the shortest distance from the pivot point to the "line of force".



Chapter 13: Rotation of a Rigid Body

The role of an axle: So long as it doesn't break, an axle will exert just the right force so that the net force on the object is zero, and the object doesn't accelerate away from the axle.

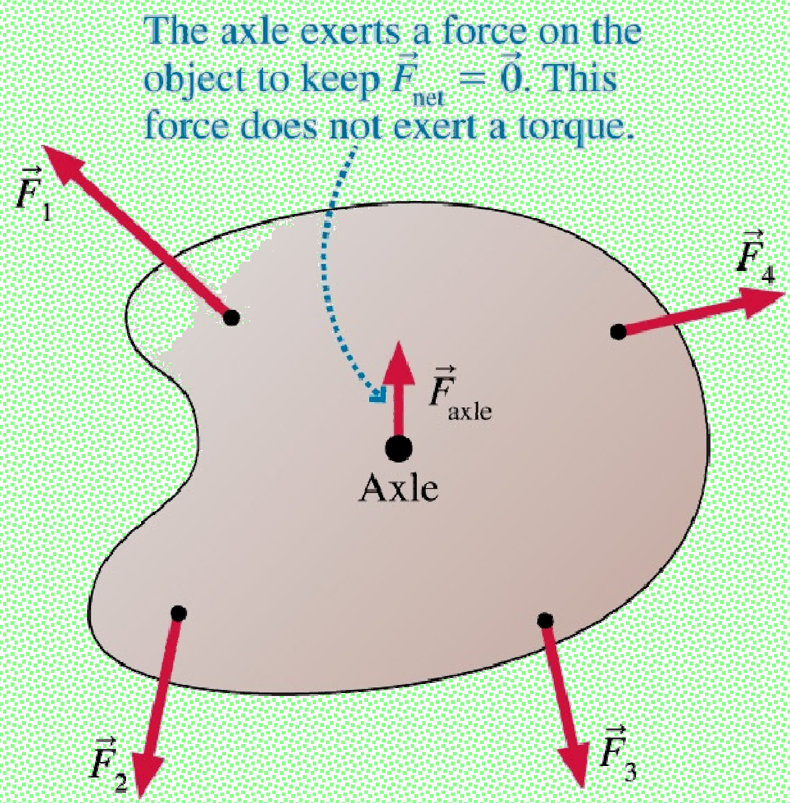
But what torque does the axle force generate?

None! It's moment arm is zero!

Thus, only the applied forces generate torque.

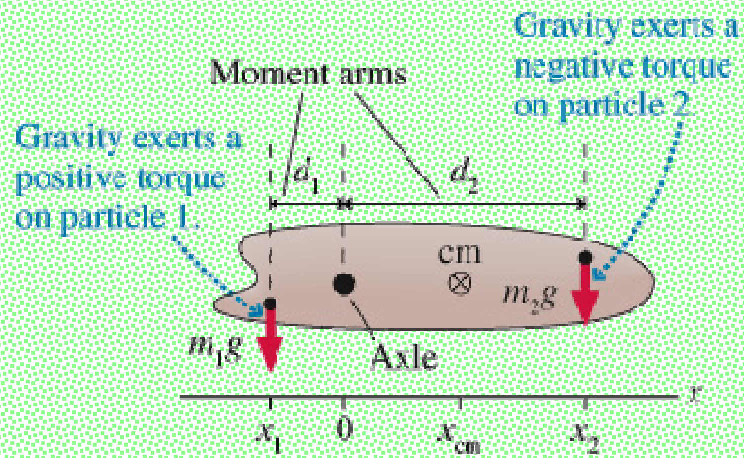
The net torque need not be zero even if the net force is.

$$\tau_{net} = \tau_1 + \tau_2 + \tau_3 + \ldots = \sum_{i=1}^{N} \tau_i$$



Chapter 13: Rotation of a Rigid Body

Torque caused by gravity



Consider a rigid body to be a collection of *N* tiny particles (all joined together), each with mass m_i, $i = 1, N$ (N very big).

Moment arm of each torque is x_i, (x-direction perpendicular to the force). Thus, the net gravitational torque is:

$$\tau_{\text{grav}} = \sum_{i=1}^{N} \tau_i = \sum_{i=1}^{N} (-m_i g x_i)$$

Chapter 13: Rotation of a Rigid Body

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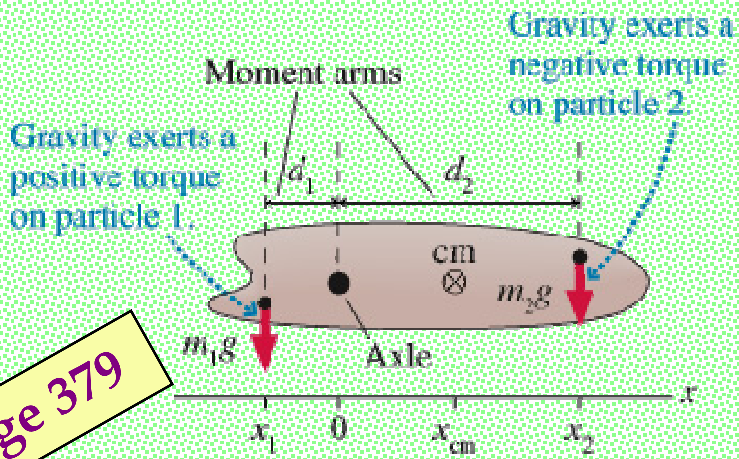
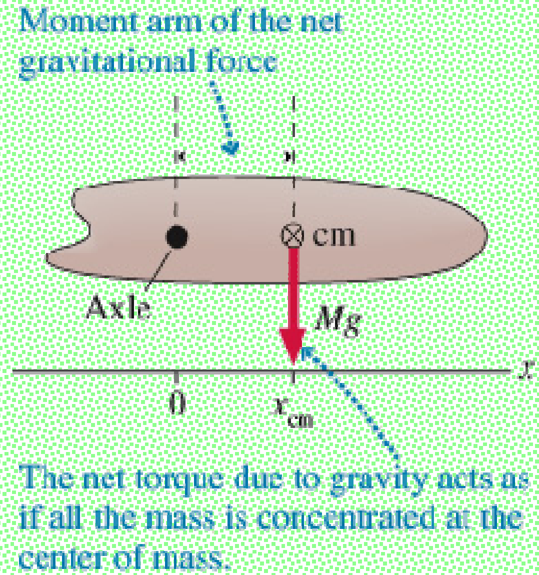
Moment arm of each torque is x_i, (x-direction perpendicular to the force). Thus, the net gravitational torque is:

see page 379

$$\tau_{\text{grav}} = \sum_{i=1}^{N} \tau_i = \sum_{i=1}^{N} (-m_i g x_i) = -Mg \underbrace{\frac{1}{M} \sum_{i=1}^{N} m_i x_i}_{x_{\text{cm}}}$$

$$\Rightarrow \quad \boxed{\tau_{\text{grav}} = -Mgx_{\text{cm}}}$$





Thus, gravitational torque acts as though all mass were concentrated at the centre of mass (measured relative to the pivot point).

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Chapter 13: Rotation of a Rigid Body

Newton's 2nd Law for rotation

A rocket of mass *m* (point particle) is attached to a rod on a pivot.

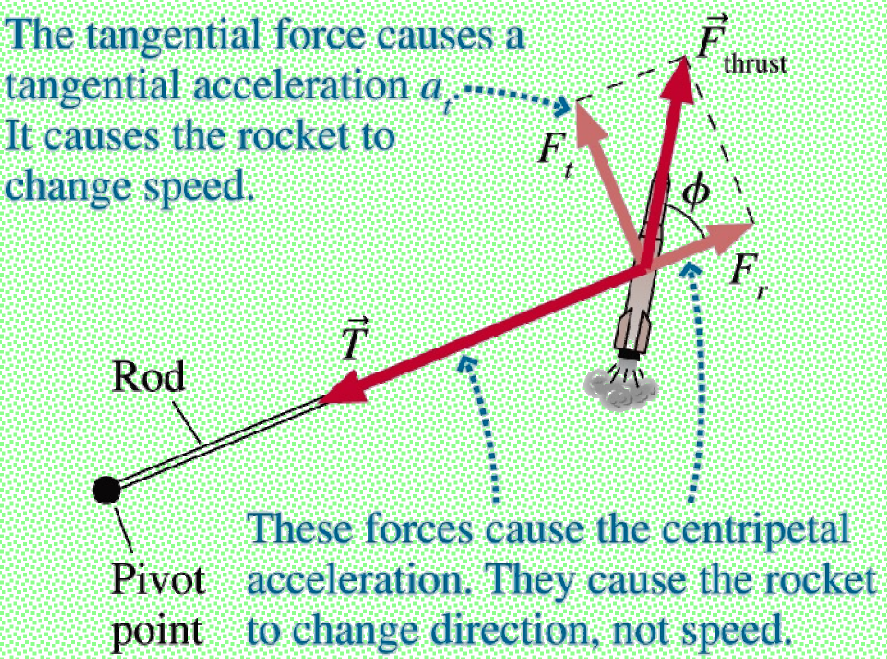
Tension in the rod counteracts the radial component of the thrust, leaving only the tangential component to cause an acceleration. In Chapter 7, we saw that a tangential force gives rise to an angular acceleration:

$$F_t = ma_t = mr\alpha$$

$$\Rightarrow \quad rF_t = \boxed{\tau = mr^2\alpha}$$

Thus, for a point mass, torque causes *angular acceleration*, just as force causes *linear acceleration*.

We now extend this idea to extended (rigid) bodies…

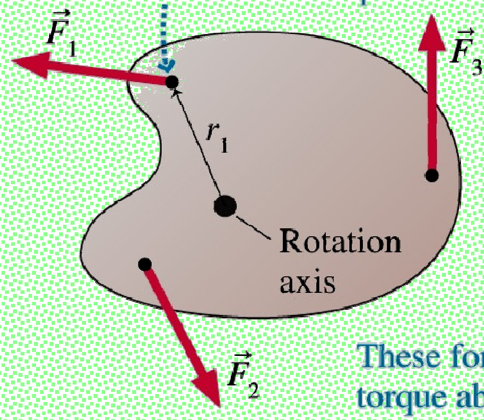


Chapter 13: Rotation of a Rigid Body

Newton's 2nd Law for rotation, continued

For an "extended" object, we do as before: Suppose the object is made up of N (very large) point masses of mass m_i, $i = 1, N$, and add up all the particle torques to get the net object torque:

$$\tau_{\text{net}} = \sum_{i=1}^{N} \tau_i = \sum_{i=1}^{N} (m_i r_i^2 \alpha) = \underbrace{\left(\sum_{i=1}^{N} m_i r_i^2\right)}_{I} \alpha$$



I = *moment of inertia* about the rotation axis.

Thus, for *rigid bodies*, we have Newton's 2nd Law for rotation:

$$\boxed{\tau_{\text{net}} = I\alpha}$$ **(compare with $F_{\text{net}} = ma$)**

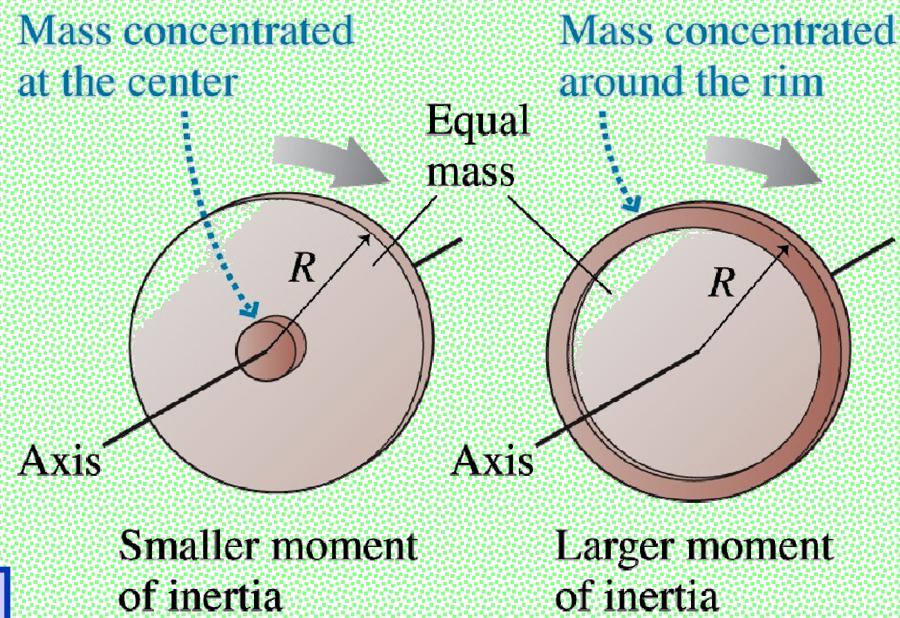
Why a "rigid body"? We need α to be the same at all points! Without a rigid body, α couldn't have been "factored out" above, and the form of Newton's 2nd Law would have been much more complicated.

Chapter 13: Rotation of a Rigid Body

The **moment of inertia** is to **angular acceleration** what **mass** (inertia) is to **linear acceleration**. **Mass** is the property of an object that resists **linear acceleration** from a **force**. The **moment if inertia** is the property of an object that resists **angular acceleration** from a **torque**.

Unlike ***m***, ***I*** isn't unique for each object: It depends on:

- the mass of the object
- distribution of mass
- location of rotation axis



moment of inertia (black tubes) demo

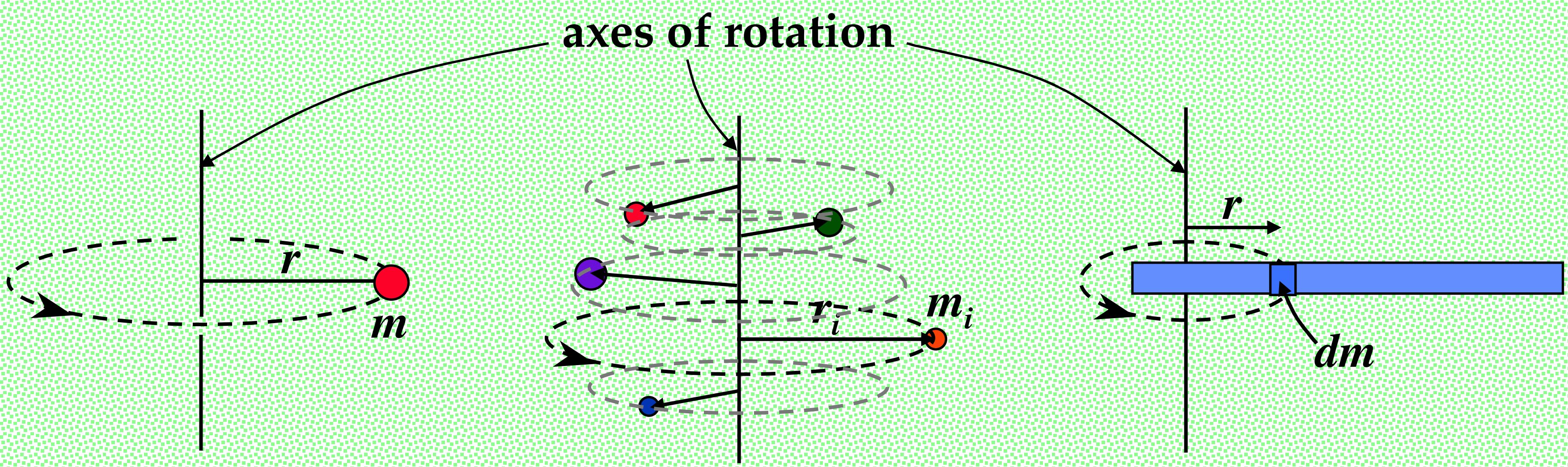
Chapter 13: Rotation of a Rigid Body

Calculating moment of inertia:

For a single point mass: $I = mr^2$ (units: kg m²)

For N point masses: $I = \sum_{i=1}^{N} m_i r_i^2$

For an extended object: $I = \lim_{\Delta m_i \to 0} \sum_{i=1}^{N} \Delta m_i r_i^2 = \int r^2 dm$



Clicker question 13.7

You swing a rock of mass 0.25 kg on the end of a rope of length 2.0 m about your head at angular speed 3 rad s^{-1}. What is moment of inertia of the rock?

a) 0.75 kg m^2

b) 1.00 kg m^2

c) 1.50 kg m^2

d) 6.00 kg m^2

Clicker question 13.7

You swing a rock of mass 0.25 kg on the end of a rope of length 2.0 m about your head at angular speed 3 rad s^{-1}. What is moment of inertia of the rock?

a) 0.75 kg m^2

b) 1.00 kg m^2 ✔

c) 1.50 kg m^2

d) 6.00 kg m^2

$I = mr^2 = (0.25)(2.0)^2 = 1.00$ kg m^2

Note that the angular speed was completely irrelevant.

Clicker question 13.8

The mass of the Earth is 6.0×10^{24} kg, and its distance from the sun is 1.5×10^{11} m. What is the moment of inertia of the earth as it orbits about the sun?

a) 9.0×10^{35} kg m^2

b) 9.0×10^{47} kg m^2

c) 1.35×10^{35} kg m^2

d) 1.35×10^{47} kg m^2

Clicker question 13.8

The mass of the Earth is 6.0×10^{24} kg, and its distance from the sun is 1.5×10^{11} m. What is the moment of inertia of the earth as it orbits about the sun?

a) 9.0×10^{35} kg m^2

b) 9.0×10^{47} kg m^2

c) 1.35×10^{35} kg m^2

d) 1.35×10^{47} kg m^2 ✔

$$I = mr^2$$

Dealing with big numbers in your head:

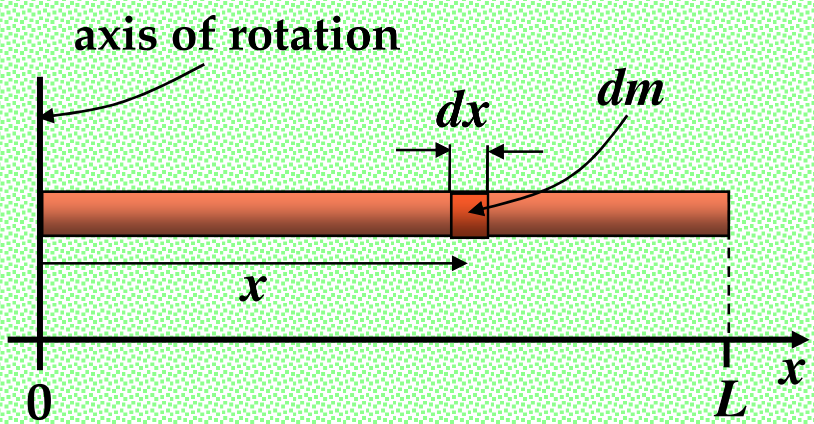
Deal with the mantissas first: You know 1.5^2 is about 2, and $2 \times 6 = 12$ ⇒ a) and b) are eliminated.

Next add the exponents: 24 from *m*, 11 twice from *r* ⇒ 24 + 11 + 11 = 46

$12 \times 10^{46} \sim 1.2 \times 10^{47}$ ⇒ d)

Chapter 13: Rotation of a Rigid Body

example: compute I for a thin rod spinning about one end.



$$I = \int_0^L x^2 dm$$

We cannot proceed until we know either x in terms of m, or m in terms of x.

Problems like this typically go as follows:

The mass per unit length of the entire rod is: $\frac{m}{L}$

The mass per unit length of the mass increment is: $\frac{dm}{dx}$

For a uniform rod, these must be the same! Thus, $dm = \frac{m}{L}dx$**, and we get:**

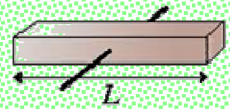
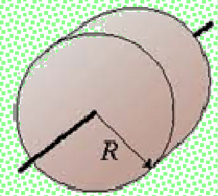
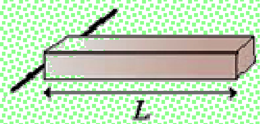
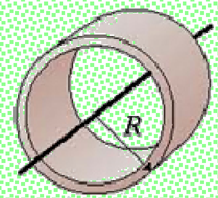
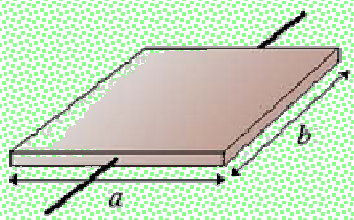
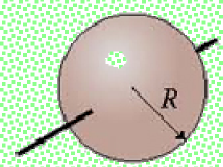
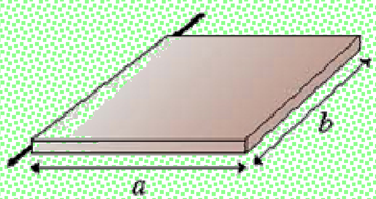
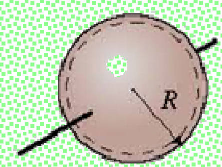
$$I = \frac{m}{L}\int_0^L x^2 dx = \frac{m}{L}\frac{x^3}{3}\Big|_0^L = \frac{m}{L}\left(\frac{L^3}{3} - 0\right) = \frac{mL^2}{3}$$

check units! kg m^2 ✔

Chapter 13: Rotation of a Rigid Body

Table 13.3, page 385

TABLE 13.3 Moments of inertia of objects with uniform density

Object and axis	Picture	I	Object and axis	Picture	I
Thin rod, about center		$\frac{1}{12}ML^2$	Cylinder or disk, about center		$\frac{1}{2}MR^2$
Thin rod, about end		$\frac{1}{3}ML^2$	Cylindrical hoop, about center		MR^2
Plane or slab, about center		$\frac{1}{12}Ma^2$	Solid sphere, about diameter		$\frac{2}{5}MR^2$
Plane or slab, about edge		$\frac{1}{3}Ma^2$	Spherical shell, about diameter		$\frac{2}{3}MR^2$

Chapter 13: Rotation of a Rigid Body

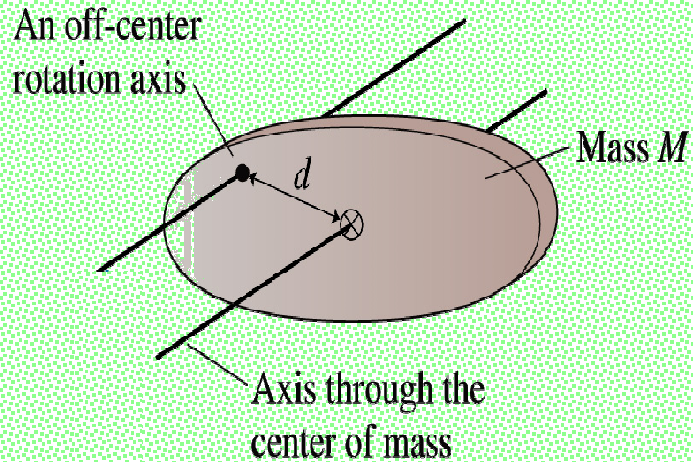
The parallel axis theorem (see page 386 for a "sorta proof"):

Let I_{cm} be the moment of inertia of a mass M about an axis that passes through the centre of mass.

Let I be the moment of inertia about an axis parallel to and at a distance d away from the first axis.

$\Rightarrow$ the two moments are related by:

$$I = I_{cm} + Md^2$$



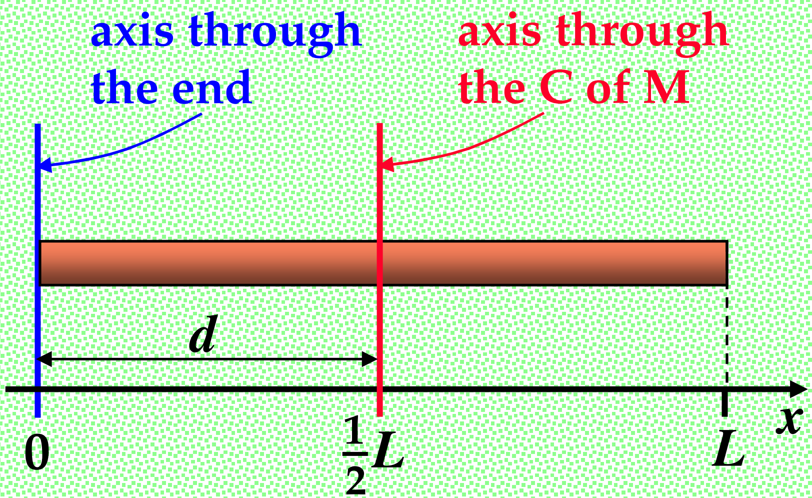
This is the parallel axis theorem.

For it to apply, the "reference" moment of inertia must be about the centre of mass, **and the two axes must be parallel!!**

Chapter 13: Rotation of a Rigid Body

example: Compute the moment of inertia of a thin rod of mass M and length L about an axis through its centre of mass, using the fact that the moment of inertia about its end is:

$$I = \tfrac{1}{3}ML^2.$$



From the parallel axis theorem:

$$I = I_{\text{cm}} + Md^2$$

Here, $d = L/2$.

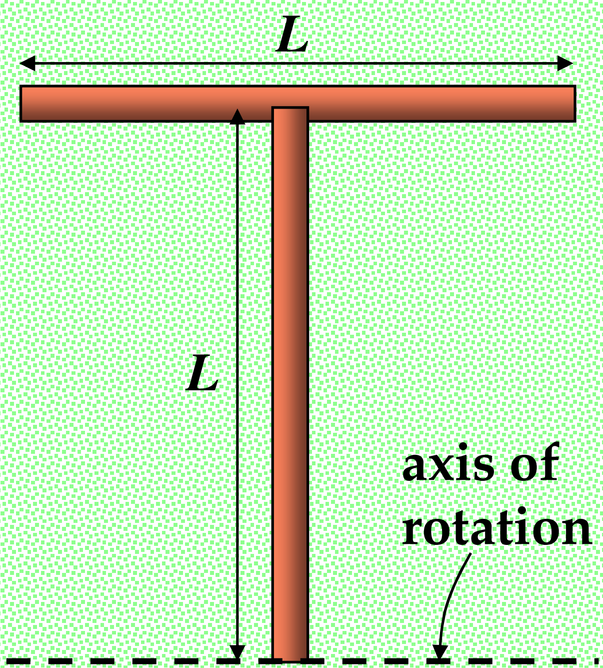
$$\Rightarrow \quad I_{\text{cm}} = I - M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3} - \frac{ML^2}{4} = \frac{ML^2}{12}$$

exercise: Try computing I_{cm} *directly from*

$$I = \int_0^L x^2 dm$$

Chapter 13: Rotation of a Rigid Body

example: A "T" is made up of two identical thin rods, as shown, each of mass M and length L. What is the moment of inertia of the "T" about an axis at its base parallel to its top?



Break the "T" up into its vertical and horizontal parts:

$$I = I_v + I_h$$

We've already done the vertical bit:

$$I_v = \frac{1}{3}ML^2$$

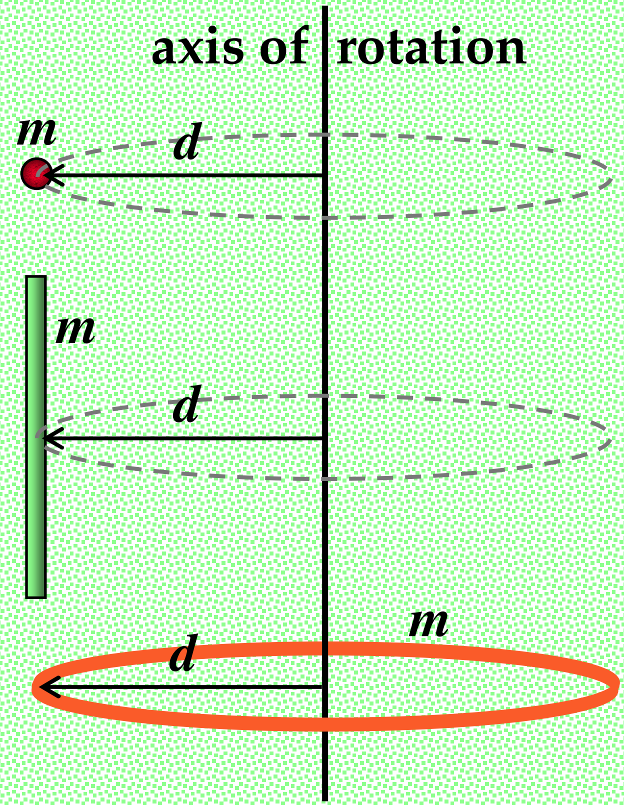
Since each bit of the horizontal rod is the same distance from the rotation axis, we can treat it like a point mass, in which case:

$$I_h = ML^2$$

$$\Rightarrow I = \frac{1}{3}ML^2 + ML^2 = \frac{4}{3}ML^2$$

Clicker question 13.9

Three objects, each of mass m, rotate about a common axis at the same distance d from the axis. Ignoring the radii of the sphere and rod and the thickness of the hoop, which has the greatest moment of inertia about the axis?

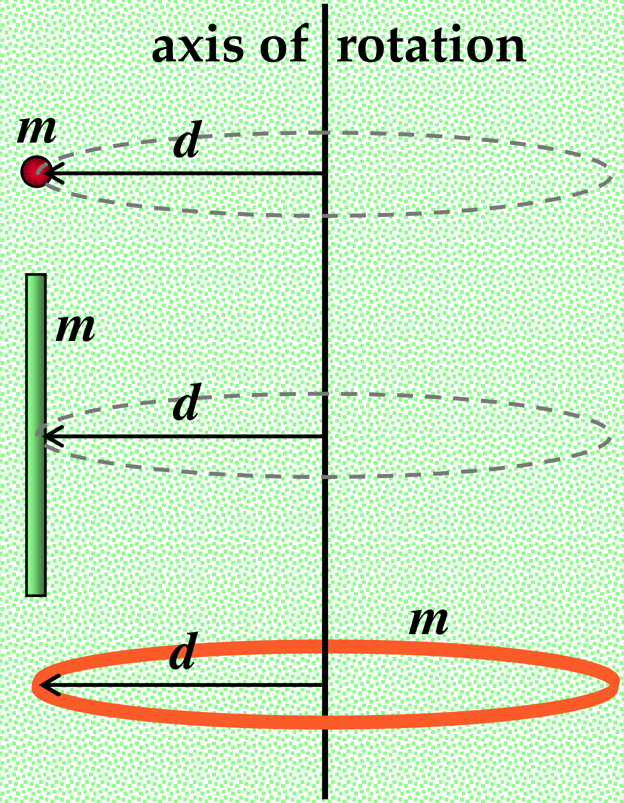


a) the sphere b) the rod c) the hoop

d) they all have the same moment of inertia

Clicker question 13.9

Three objects, each of mass m, rotate about a common axis at the same distance d from the axis. Ignoring the radii of the sphere and rod and the thickness of the hoop, which has the greatest moment of inertia about the axis?



a) the sphere b) the rod c) the hoop

d) they all have the same moment of inertia ✔

Chapter 13: Rotation of a Rigid Body

13.5 Rotation about a fixed axis

Problem solving strategy (page 387)

1. Model object as a simple shape
2. Visualise: draw a pictorial representation, FBD, *etc*.
   - set a coordinate system
   - identify a rotation axis
   - identify forces and their distances from the rotation axis
   - identify torques and their signs
3. Solve: mathematical representation ($\tau_{net} = I\alpha$)
   - look up I and/or use parallel axis theorem
   - use rotational kinematics to find ω and/or $\Delta\theta$

Chapter 13: Rotation of a Rigid Body

example: A wheel of mass $M = 5.0$ kg and radius $r = 0.050$ m has an axis of rotation located $d = r/2$ from the centre. A vertical tension $T = 100$ N is exerted at the rim of the wheel, as shown. A pin holding the wheel in place is removed at $t = 0$. Find α the instant after the pin is removed.

Model the wheel as a uniform disc. Gravity exerts a torque because the axis is off-centre.

Visualise: Diagram shows the forces, distances from axis, *etc*.

Solve: $d = \frac{r}{2}$ = magnitude of both $\vec{r}_T$ and $\vec{r}_g$

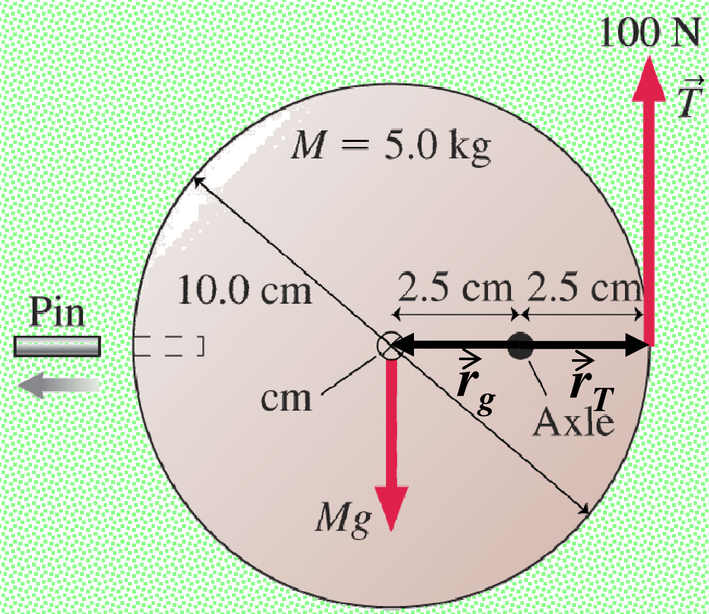
$$\Rightarrow \quad \tau_g = Mgd = \tfrac{1}{2}Mgr \qquad \tau_T = Td = \tfrac{1}{2}Tr$$

both torques act ccw $\Rightarrow$ both are positive.

$$I = \tfrac{1}{2}Mr^2 + Md^2 = \tfrac{1}{2}Mr^2 + \tfrac{1}{4}Mr^2 = \tfrac{3}{4}Mr^2$$

$$\tau_{net} = \tau_g + \tau_T = 3.725 \text{ Nm}; \; I = 9.375 \times 10^{-3} \text{ kg m}^2$$

$$\Rightarrow \quad \alpha = \frac{\tau_{net}}{I} = 397 \text{ rad s}^{-2}$$

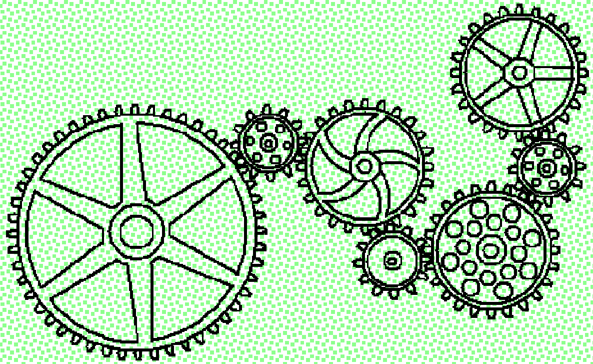


Chapter 13: Rotation of a Rigid Body

Ropes, pulleys, and gears…

Consider a rotating object connected to another (possibly rotating) object:

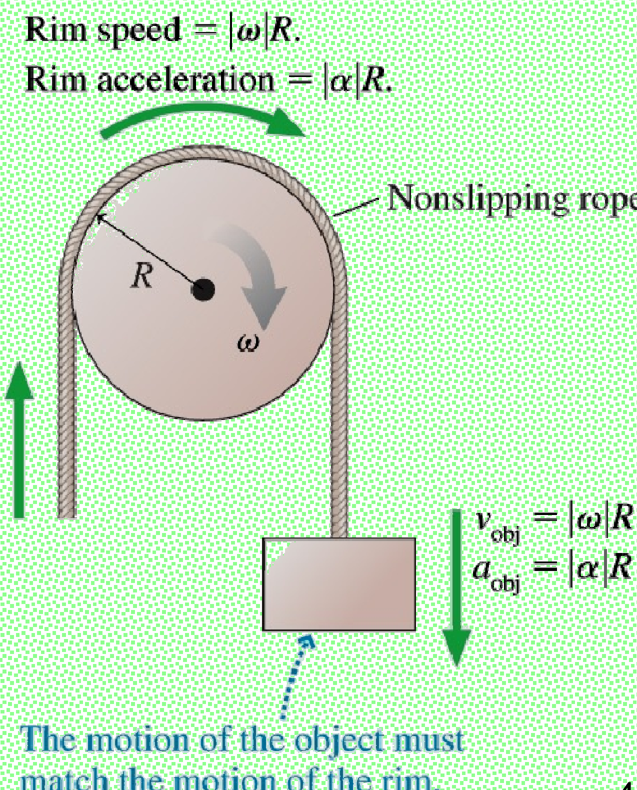
- in direct contact, such as gears;
- *via* ropes, belts, *etc*.



So long as touching objects move **without slipping**, then

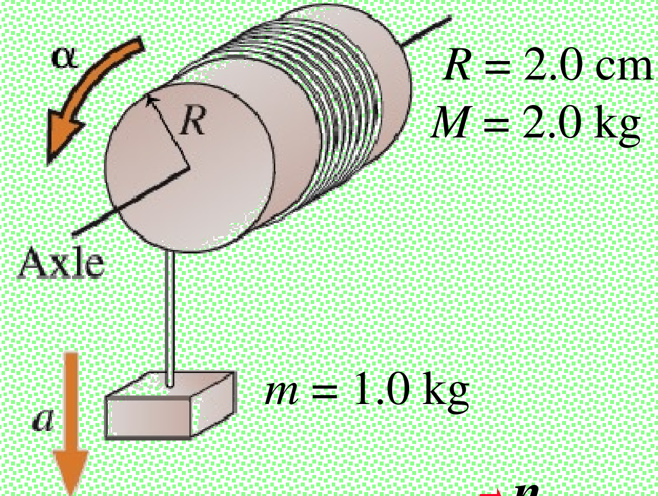
gears: Points in contact must have the same tangential speed & acceleration;

rope on a pulley: Rope's linear speed and acceleration must equal the tangential speed and acceleration at rim of pulley.



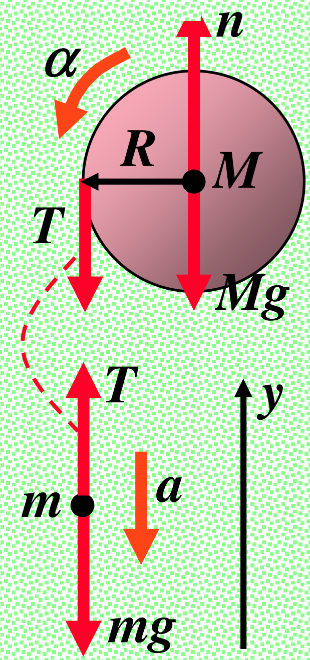
Chapter 13: Rotation of a Rigid Body

example: A mass ***m*** = 1.0 kg hangs on a massless string wrapped around a cylinder of mass ***M*** = 2.0 kg, radius ***R*** = 2.0 cm. The cylinder rotates without friction on a horizontal axis through its axis of symmetry.



What is the acceleration of ***m***?

Model: point mass for ***m***, rigid body for ***M***, no-slip condition for rope. Thus, $a = \alpha R$.



Visualise : For ***M***, the normal force exerted by the axle, ***n***, exactly balances ***Mg*** + ***T***, and ***M*** has no linear acceleration.

Chapter 13: Rotation of a Rigid Body

example, continued… from previous slide: $a = \alpha R$

Visualise: $\boldsymbol{n}$ and $\boldsymbol{Mg}$ act through the pivot, and generate no torque. $\boldsymbol{T}$ acts tangentially to the rim and generates the torque:

$$\tau = RT(\sin 90°) = RT \quad (\text{ccw} \Rightarrow \text{positive}).$$

Solve: Newton's 2nd Law for rotation ($\boldsymbol{M}$):

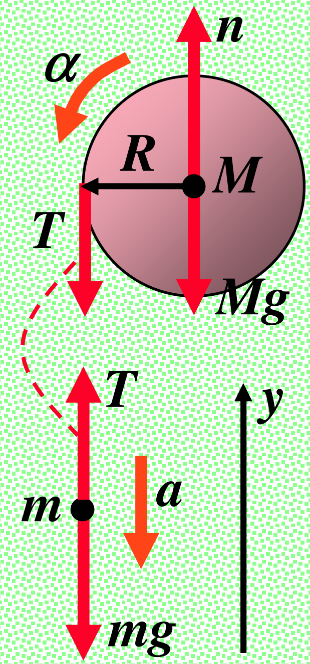
$$\tau = I\alpha \;\Rightarrow\; RT = \tfrac{1}{2}MR^2\frac{a}{R} \;\Rightarrow\; T = \tfrac{1}{2}Ma \qquad (1)$$

Newton's 2nd Law for $\boldsymbol{m}$:

$$T - mg = -ma \;\Rightarrow\; T = mg - ma \qquad (2)$$

Compare (1) and (2) $\Rightarrow \tfrac{1}{2}Ma + ma = mg$

$$\Rightarrow \quad a = \frac{mg}{\tfrac{1}{2}M + m} = \frac{g}{2} = 4.9\ \text{ms}^{-2}$$



Chapter 13: Rotation of a Rigid Body

13.6 Rigid-body equilibrium

In engineering design, the concept of equilibrium is critical. For a bridge not to be in equilibrium is to invite disaster!

A rigid body is in equilibrium (*i.e.,* won't move!) if

$$\vec{F}_{\text{net}} = 0 \quad \textbf{and} \quad \vec{\tau}_{\text{net}} = 0$$

regardless of which axis you choose!

These are vector equations, each with three components,

⇒ six equations in all! ☹

We shall limit ourselves to problems in which all forces lie in the ***x-y*** plane, and all torques are about axes perpendicular to the ***x-y*** plane (*i.e.,* in the z-direction).

⇒ three equations, one each for F_x, F_y, and τ ☺

Chapter 13: Rotation of a Rigid Body

Problem-solving strategy for equilibrium problems (page 390):

Model object as a simple shape.

Visualize: Draw pictorial representation and FBD.

- **Pick any point you want as a pivot point.** *The algebra is much easier if you pick a point through which most of the unknown forces act!*
- **Determine the moment arms of all forces about your pivot point.**
- **Determine the sign of each torque about your pivot point.**
- **If the direction of an unknown force is also unknown, represent it as *two* perpendicular forces: F_x and F_y acting at the same point.**

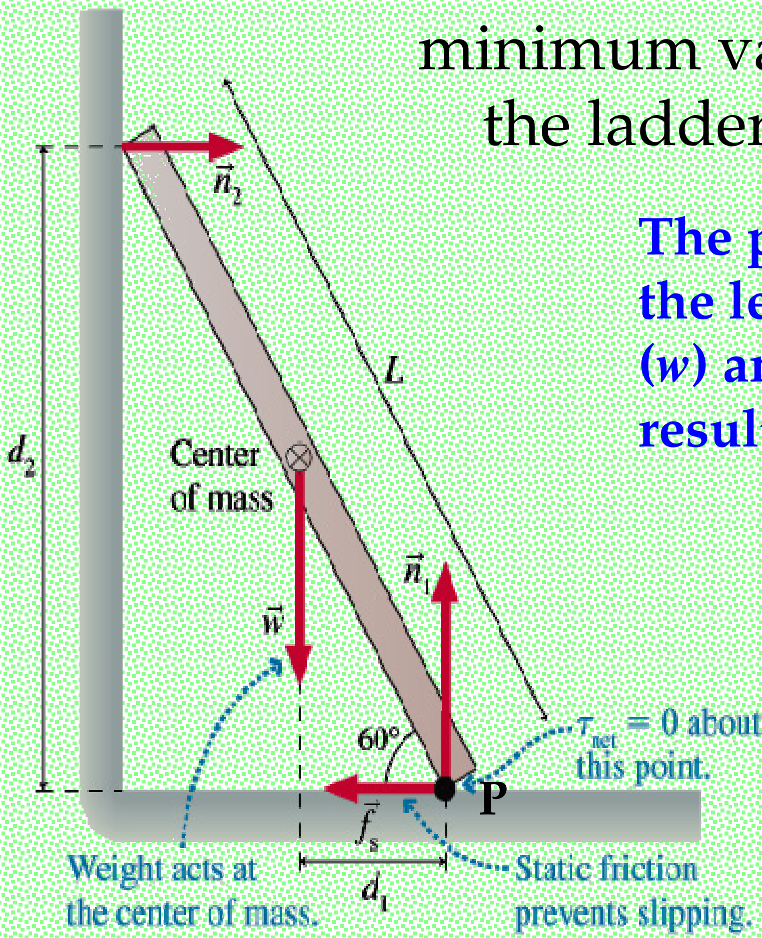
Solve: No net forces and no net torque about *any* pivot point.

- $\tau_{net} = 0$, $F_{net,x} = 0$, **and** $F_{net,y} = 0$
- **Solve these three equations for any unknown forces, distances, *etc*.**

Assess: Is the answer reasonable; does it answer the question?

Chapter 13: Rotation of a Rigid Body

example: Will the ladder slip? (page 391) A ladder rests against a frictionless wall at an angle $\theta = 60°$. What is the minimum value of μ_s between the ground and the ladder to prevent slipping?



The problem says nothing about the mass nor the length of the ladder. We may introduce *m* (*w*) and *L* as *interim* quantities, but our final result must be independent of them!

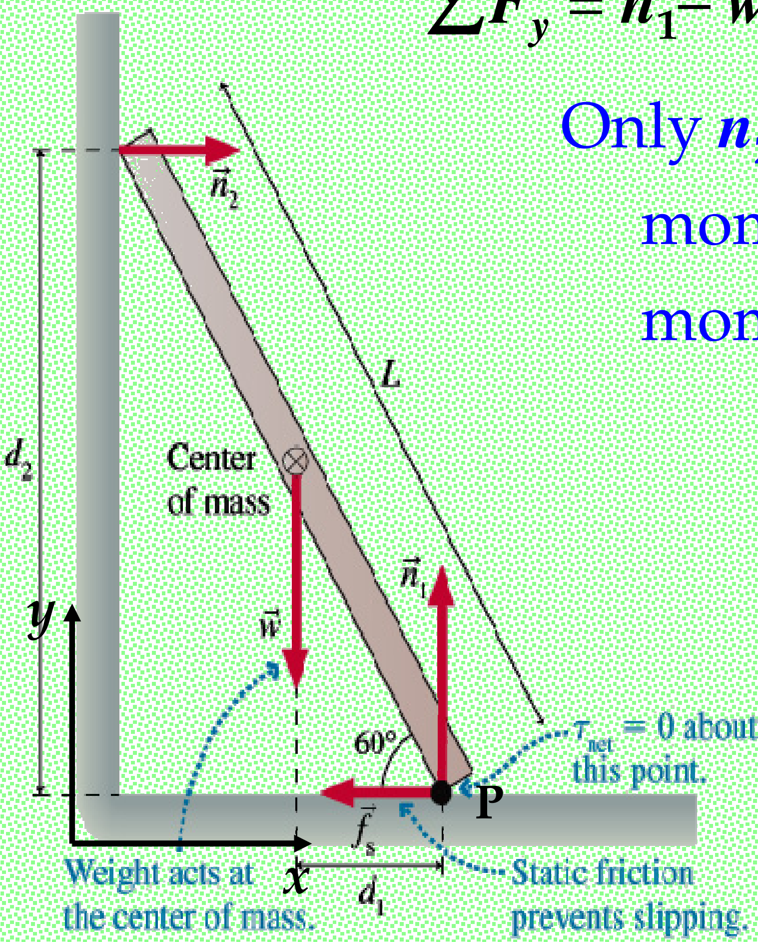
Choose P as the pivot point. It has the most number of forces acting through it, which will reduce the number of torques we have to identify (zero moment arms!)

Since we seek the *minimum* μ_s, we can set $f_s = \mu_s n_1$.

Chapter 13: Rotation of a Rigid Body

$$\sum F_x = n_2 - f_s = 0 \quad \Rightarrow \quad n_2 = \mu_s n_1 \qquad (1)$$

$$\sum F_y = n_1 - w = 0 \quad \Rightarrow \quad w = n_1 \qquad (2)$$



Only n_2 and w generate torques about P:

moment arm of n_2 is: $d_2 = L\sin\theta$

moment arm of w is: $d_1 = \frac{1}{2}L\cos\theta$

$$\sum \tau = \tfrac{1}{2}w\cancel{L}\cos\theta - n_2\cancel{L}\sin\theta = 0 \qquad (3)$$

(torque caused by w ccw, n_2 cw)

Substitute (1) and (2) into (3) $\Rightarrow$

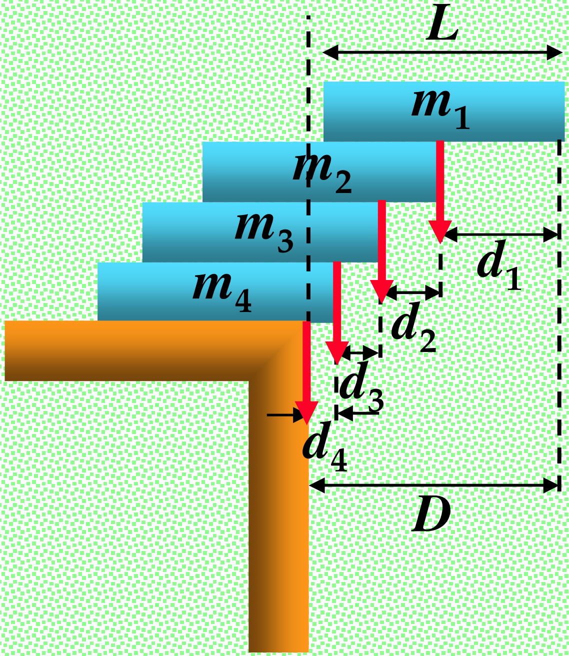
$$\tfrac{1}{2}\cancel{n_1}\cos\theta - \mu_s\cancel{n_1}\sin\theta = 0$$

$$\Rightarrow \mu_s = \frac{\cot\theta}{2} = 0.29.$$

Chapter 13: Rotation of a Rigid Body

Example: The Tower of Lyre, or the "great pub bet"

At the pub you bet your friend the next round that you can stack four blocks (they can be coasters) over the edge of the table such that the top block is fully over the edge of the table ($D > L$). After he tries for a few minutes in futility, you, the keen physics student, just "stack 'em up"! How?



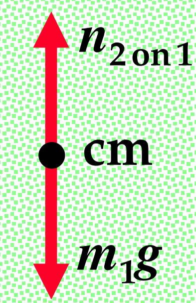
Let $m_1 = m_2 = m_3 = m_4 = m$

1. Key: All masses are on the *verge* of tipping. Thus, the normal force m_i exerts on m_{i+1} right below it is applied right at the edge of m_{i+1}.

So to start, we can deduce that m_1 can balance as much as $d_1 = L/2$ over the edge of m_2.

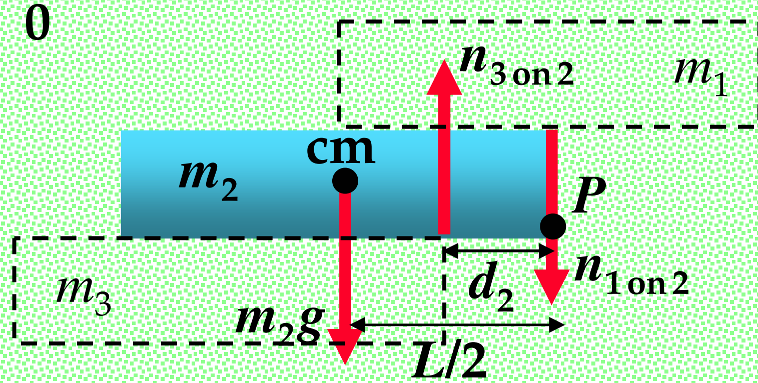
2. FBD for m_1:

$$n_{2\,\mathrm{on}\,1} - m_1 g = 0 \;\Rightarrow\; n_{2\,\mathrm{on}\,1} = mg$$



Chapter 13: Rotation of a Rigid Body

3. FBD for m_2: $\sum F_y = n_{3\text{ on }2} - n_{1\text{ on }2} - m_2 g = 0$



$n_{1\text{ on }2}$ and $n_{2\text{ on }1}$ form an action-reaction pair.

$\Rightarrow \quad n_{1\text{ on }2} = n_{2\text{ on }1} = mg \quad \Rightarrow \quad n_{3\text{ on }2} = 2mg$

No surprise. Next, examine the torques.

Choose point P as our pivot (any point will do).
Relative to P, only $m_2 g$ and $n_{3\text{ on }2}$ generate torques:

$m_2 g$ generates a ccw torque about P; its moment arm is $L/2$

$n_{3\text{ on }2}$ generates a cw torque about P; its moment arm is d_2

$$\sum \tau = \frac{L}{2} m_2 g - d_2 n_{3\text{ on }2} = 0 \quad \Rightarrow \quad 2mg d_2 = \frac{L}{2} mg \quad \Rightarrow \quad d_2 = L/4$$

4. Repeat (try it!) for m_3 ($d_3 = L/6$) and m_4 ($d_4 = L/8$; see the pattern?)

$$\Rightarrow \quad D = d_1 + d_2 + d_3 + d_4 = L\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8}\right) = \frac{25}{24} L \;>\; L$$

Chapter 13: Rotation of a Rigid Body

Two last points on the tower of Lyre…

1. If you plan to try this as a pub bet, best to do it with *five* "bricks". With five, $\boldsymbol{D} = 137\boldsymbol{L}/120$, giving you $17\boldsymbol{L}/120$ or 14%of $\boldsymbol{L}$ to play with. With just four bricks, you only have 1/24 (< 5%) to play with***, and this may require more accuracy than one might have after a couple of pints***…

2. You may know that the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots$ does ***not*** converge! This means that with enough bricks, you could build a bridge from the table all the way across the country, with only one support!

Although even with 1 ***billion*** bricks (stack height ~ 13% of the way to the moon!), you are still under 11 $\boldsymbol{L}$ beyond the edge of the table, so you better get a lot of bricks!

ended here, 25/11/08

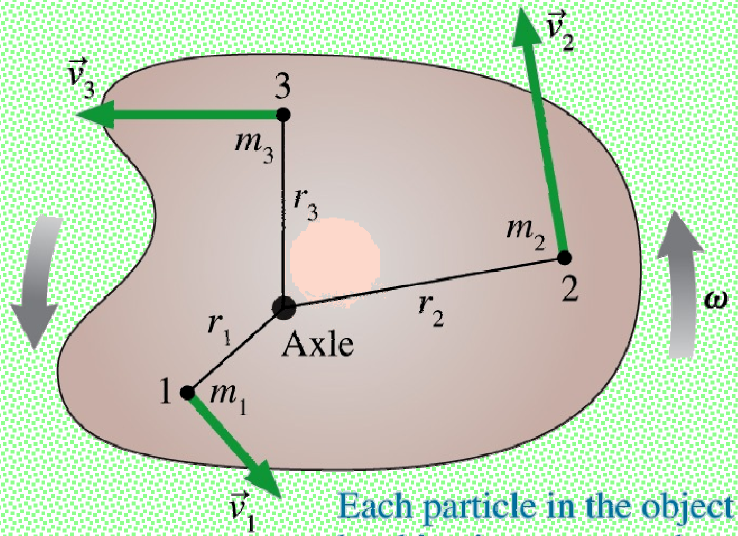
13.7 Rotational Kinetic Energy

Every point in a rotating solid body **rotates with the same angular speed**, ω, **but with a different linear speed,** v. So how do we compute the kinetic energy?

As we've done before, break the object into N small masses, m_i, each rotating about the axle with speed $v_i = r_i\omega$. Thus, K_{rot} is given by:

$$K_{rot} = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i r_i^2 \omega^2$$

$$= \frac{1}{2} \underbrace{\left(\sum_{i=1}^{N} m_i r_i^2 \right)}_{I} \omega^2 = \frac{1}{2} I \omega^2$$



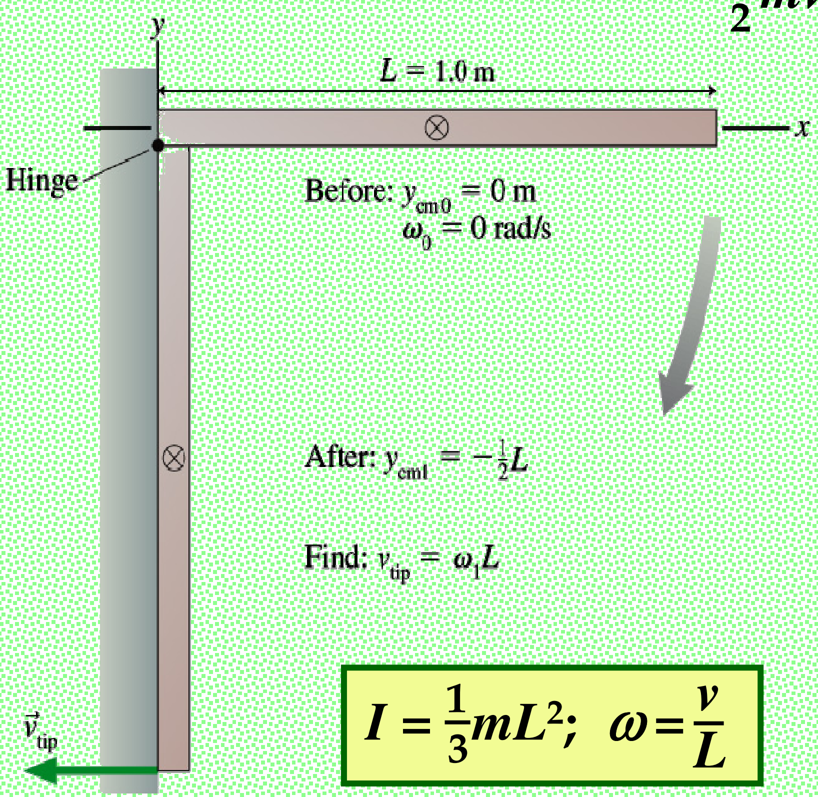
Each particle in the object has kinetic energy as the object rotates.

Chapter 13: Rotation of a Rigid Body

Rotational Kinetic energy, continued…

With the analogy between translational and rotational variables we have built up so far, including $m \leftrightarrow I$ and $v \leftrightarrow \omega$, we might have guessed that:

$$\frac{1}{2}mv^2 \leftrightarrow \frac{1}{2}I\omega^2$$



$$I = \frac{1}{3}mL^2; \ \ \omega = \frac{v}{L}$$

***example*: conservation of mechanical energy: $\Delta K + \Delta U = 0$**

A hinged (frictionless) horizontal rod ($L = 1.0$ m) is dropped from rest. What is its speed as it hits the wall?

ΔU = change in potential energy of the centre of mass = $-mgL/2$.

$$\Delta K = \frac{1}{2}I\omega^2 = -\Delta U = \frac{1}{2}mgL$$

$$\Rightarrow \ \frac{1}{3}mL^2\frac{v^2}{L^2} = mgL \ \Rightarrow \ v = \sqrt{3gL} = 5.4 \text{ ms}^{-1}$$

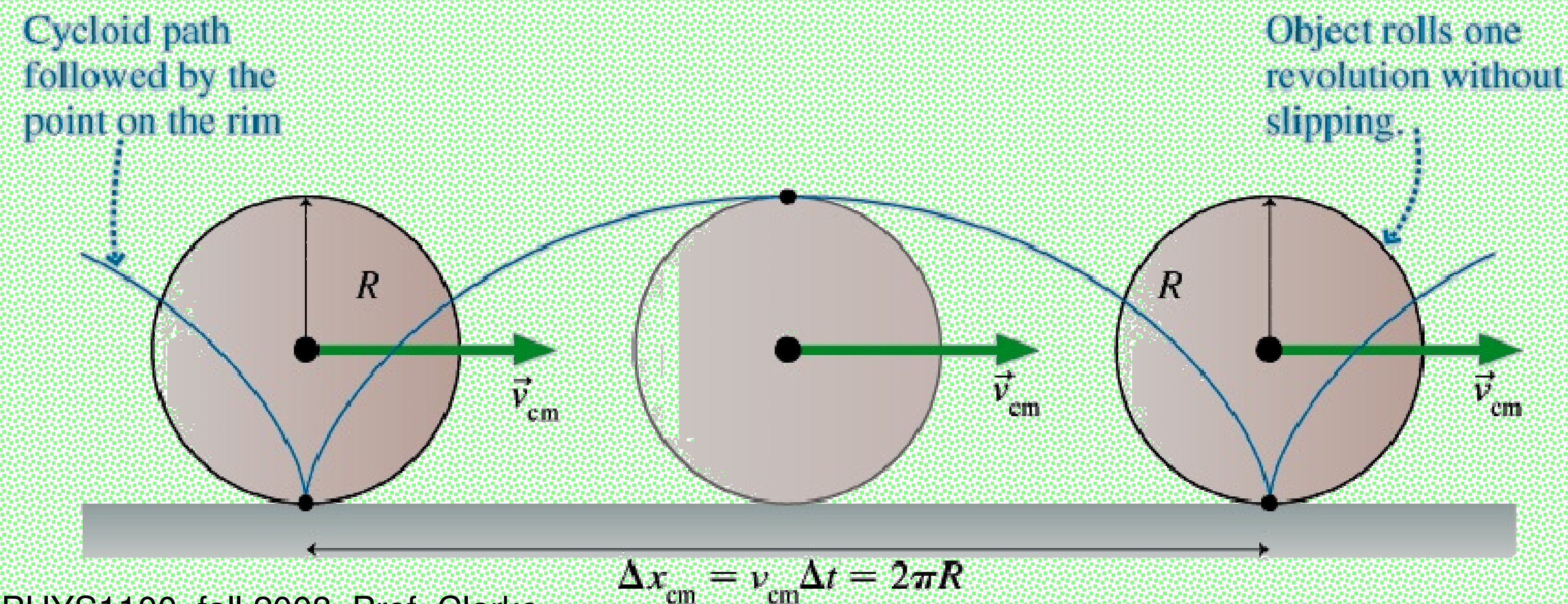
Chapter 13: Rotation of a Rigid Body

13.8 Rolling motion (combination of translation and rotation)

To "roll" means not to slip. After one revolution, centre of mass moves forward by one circumference:

$$\Delta x_{cm} = v_{cm}T = 2\pi R \quad \Rightarrow \quad v_{cm} = \frac{2\pi}{T}R = \omega R$$

$v_{cm} = \omega R$ is the rolling constraint. It links translational motion (v_{cm}) with rotation (ω), and is analogous to the no-slip condition for ropes/pulleys.



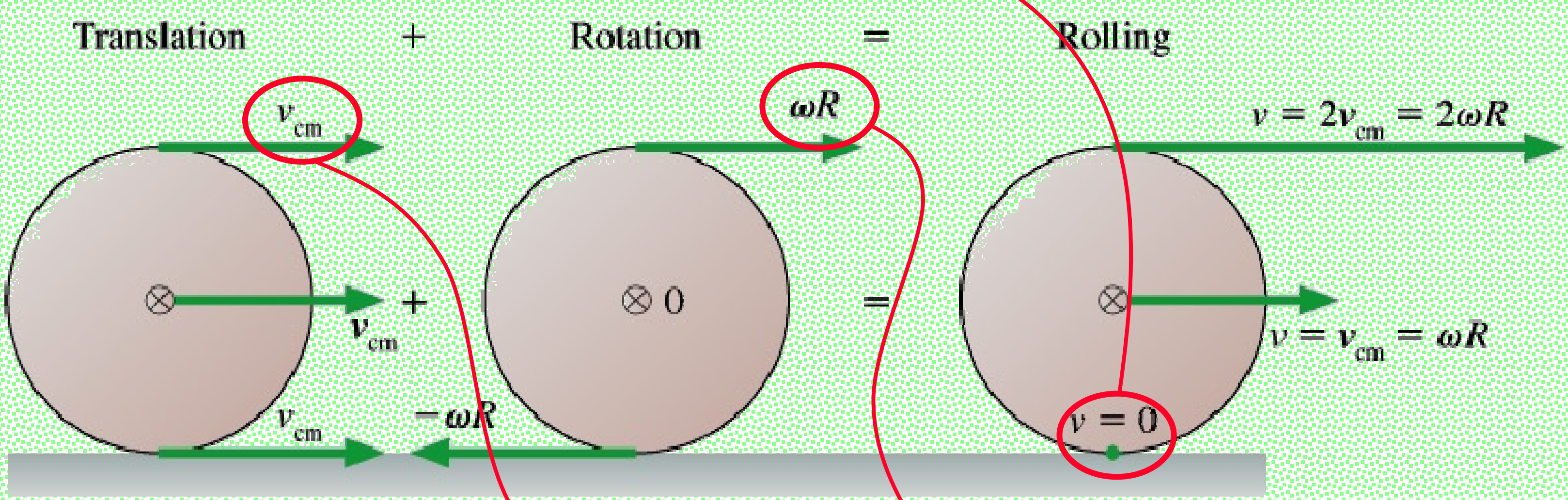
Chapter 13: Rotation of a Rigid Body

Did you know…

…the point where the tire touches the road (not skidding!) is **momentarily at rest** relative to the road **no matter how fast the car is going**?

Thus, the friction between the tire and the road is <u>static friction</u>, not kinetic friction!

This is why you have less control of the car when the tires are slipping (skidding): kinetic friction is weaker than static friction.



$v_{cm} = \omega R$ for no slipping

Chapter 13: Rotation of a Rigid Body

example: "School Fred" **Will the spool of thread move to the right or left? What is its acceleration?**

We *guess* $a > 0 \Rightarrow \alpha < 0$. This means f_s points to left (otherwise bottom slips to the right).

Forces: $F - f_s = ma \quad \Rightarrow \quad f_s = F - ma \qquad (1)$

Torques about A: $-f_s R + Fr = -I\alpha \qquad (2)$

no slipping $\Rightarrow \quad \alpha = \dfrac{a}{R} \qquad (3)$

$m = 0.10$ kg
$R = 0.030$ m
$r = 0.010$ m
$F = 0.010$ N
$I = \frac{9}{20}mR^2$
$= 5.0 \times 10^{-5}$ kg m^2

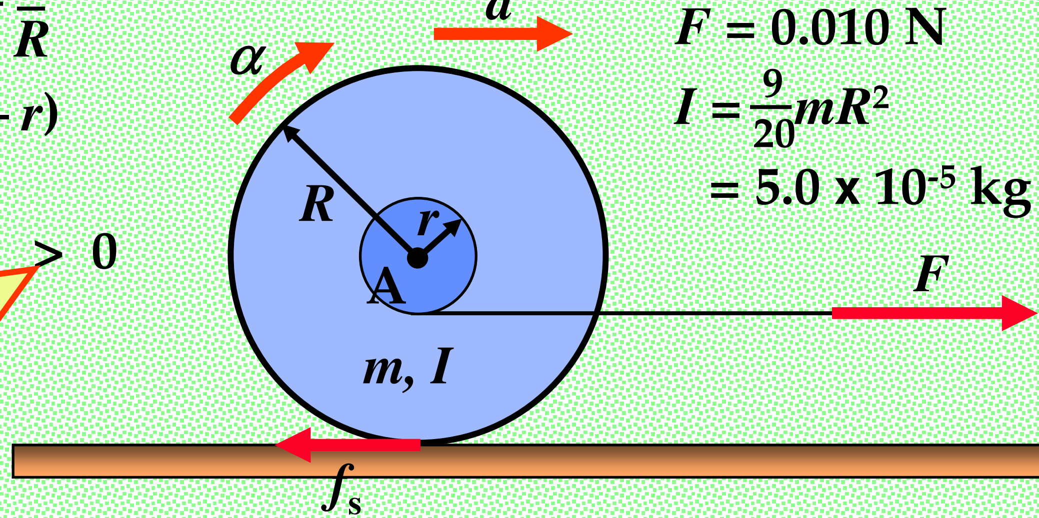
Substitute (1) and (3) into (2) $\Rightarrow$

$$R(-F + ma) + Fr = -\frac{9}{20}mR^2\frac{a}{R}$$

$$\Rightarrow \quad a\left(mR + \frac{9}{20}mR\right) = F(R - r)$$

$$a = \frac{F(R-r)}{\frac{29}{20}mR} = 0.460 \text{ ms}^{-2} > 0$$

$\Rightarrow$ our guess was right!



Chapter 13: Rotation of a Rigid Body

The energy equation, revisited.

Kinetic energy of rolling motion is ***the sum of the translational kinetic energy of the centre of mass and the rotational kinetic energy about the centre of mass*** (see page 395 for proof). Thus:

$$K = K_{cm} + K_{rot} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$

and the revised energy equation (Chapter 11) reads:

$$\Delta E_{sys} = \Delta K_{cm} + \Delta K_{rot} + \Delta U + \Delta E_{th} = W_{ext}$$

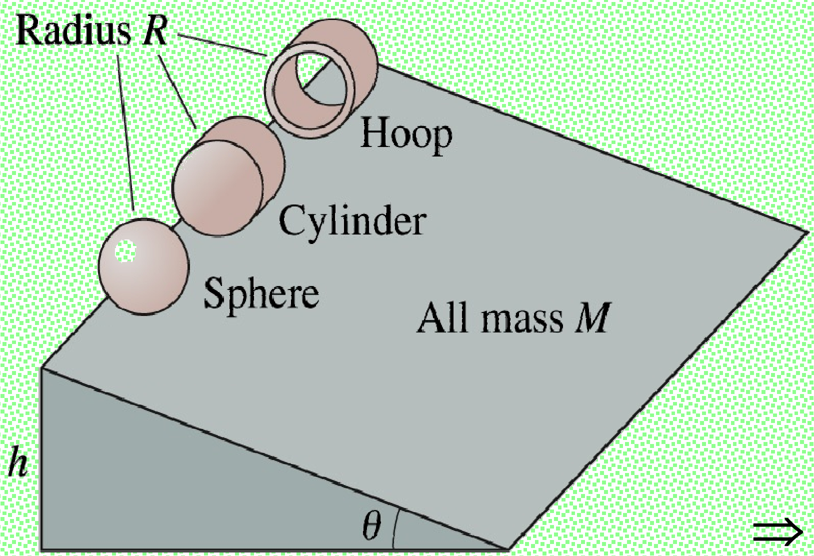
where ΔE_{sys} is the change in the total energy of the system, ΔU is the change in potential energy of the system, ΔE_{th} is the change in thermal energy, and W_{ext} is the work done by all forces external to the system.

Chapter 13: Rotation of a Rigid Body

Example: A solid sphere, a solid cylinder, and a hoop (hollow cylinder) roll down an incline. If each have the same mass and radius, which gets to the bottom first?

For any object: $\Delta K = K_f - K_i = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$; $\Delta U = -Mgh$.

No friction $\Rightarrow \Delta E_{th} = 0$; **no external forces** $\Rightarrow W_{ext} = 0$.



Let $I = cMR^2$. **Since** $\omega = \frac{v_{cm}}{R}$ **(no slip),**

$$I_{cm}\omega^2 = cMv_{cm}^2 \Rightarrow \Delta K = \frac{1}{2}Mv_{cm}^2(1 + c)$$

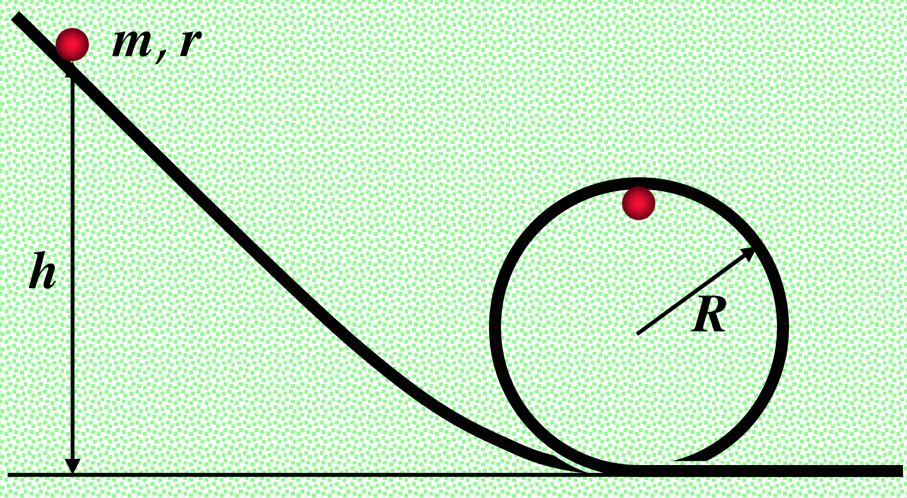
Thus, $\Delta K = -\Delta U \Rightarrow \frac{1}{2}\cancel{M}v_{cm}^2(1 + c) = \cancel{M}gh$

$$\Rightarrow v_{cm} = \sqrt{\frac{2gh}{1 + c}}$$

$\Rightarrow v_{cm}$ **is greatest for smallest** c. **For a solid sphere,** $c = \frac{2}{5}$, **for a solid cylinder,** $c = \frac{1}{2}$, **and for a hoop,** $c = 1$. **Thus the sphere gets down the ramp first.**

Chapter 13: Rotation of a Rigid Body

example: The "loop-the-loop" revisited: A solid sphere of radius r rolls without slipping down a ramp and takes a "loop-the-loop" of radius $R \gg r$. **At what minimum height, h, must the sphere be released in order for it to still make it to the top of the loop?**



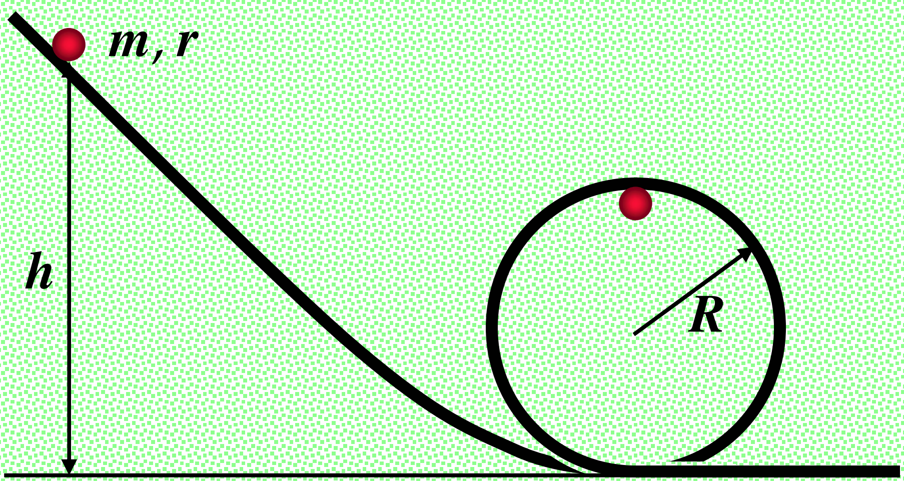
This is a **GREAT** problem. It's one of those pivotal problems in first year physics that, *when you can do it*, lets you know you've "arrived", at least to the first station on the track to becoming a physicist or an engineer. Expect one like it on the exam!

Chapter 13: Rotation of a Rigid Body

example: The "loop-the-loop" revisited…

Model: We must treat the sphere as a rigid body, since some of the kinetic energy is "used up" in rotation. The fact that $R \gg r$ is not telling us to treat the sphere as a point particle; it's needed later when calculating ΔU.

Visualise: To "barely" make it to the top of the loop does not mean the sphere is at rest there! In fact, we know from Chapter 7 that the sphere has a "critical speed" at the top of the loop, which we'll recompute here.



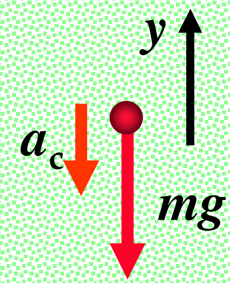
Barely making it to the top means:

1. the sphere is still traveling in a circular path (as opposed to the parabolic trajectory it would have had it left the track);

2. the normal force exerted on the sphere by the track is zero there.

Chapter 13: Rotation of a Rigid Body

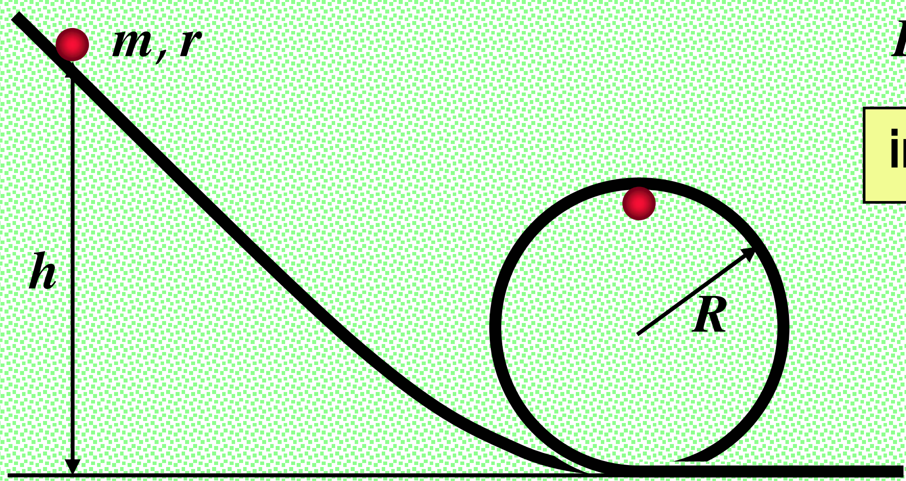
example: The "loop-the-loop" revisited…



Solve: 1. What is the speed at the top of the loop?

From the FBD, we have $-mg = -ma_c = -m\frac{v^2}{R} \;\Rightarrow\; v^2 = gR$

2. Use the energy equation. Let the system be sphere + track + earth. With no dissipation and no external forces, conserve mechanical energy.



$E_{M,i} = U_i + K_i = mgh$

$E_{M,f} = U_f + K_f = mg(2R) + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

in fact, $U_f = mg(2R-r)$, but we're told $R \gg r$

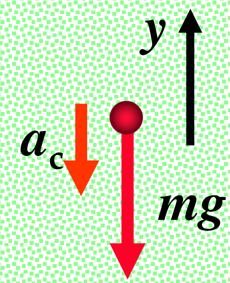
$I = \frac{2}{5}mr^2, \;\; \omega = \frac{v}{r}$ ← for no slipping

$\Rightarrow \frac{1}{2}I\omega^2 = \frac{1}{5}mv^2$

$\Rightarrow E_{M,f} = 2mgR + \frac{7}{10}mv^2$ ← $\frac{1}{2} + \frac{1}{5} = \frac{7}{10}$

Chapter 13: Rotation of a Rigid Body

example: The "loop-the-loop" revisited…



Solve: 1. What is the speed at the top of the loop?

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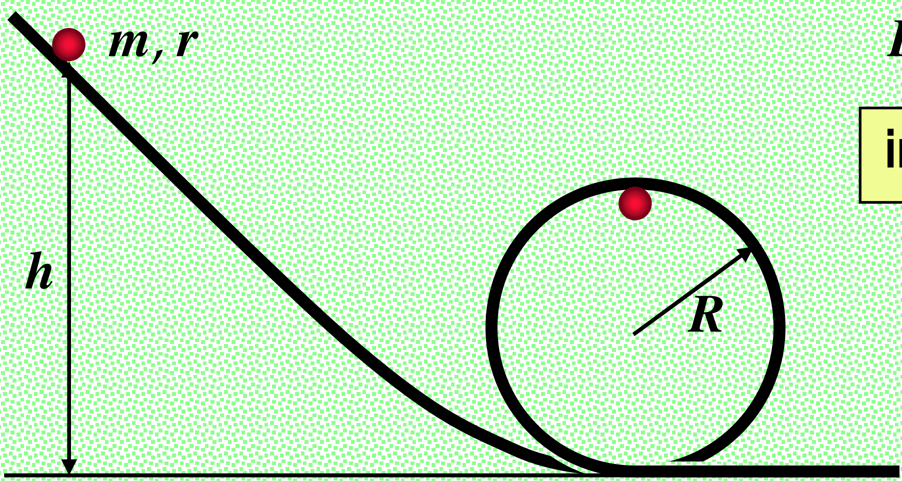
2. Use the energy equation. Let system be sphere + track + earth. With no dissipation and no external forces, conserve mechanical energy.



$$E_{M,i} = U_i + K_i = mgh$$

$$E_{M,f} = U_f + K_f = mg(2R) + \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

in fact, $U_f = mg(2R\text{-}r)$, but we're told $R \gg r$

$$I = \tfrac{2}{5}mr^2,\quad \omega = \frac{v}{r}$$

$$\Rightarrow \tfrac{1}{2}I\omega^2 = \tfrac{1}{5}mv^2$$

$$\Rightarrow E_{M,f} = 2mgR + \tfrac{7}{10}mv^2 = 2.7mgR$$

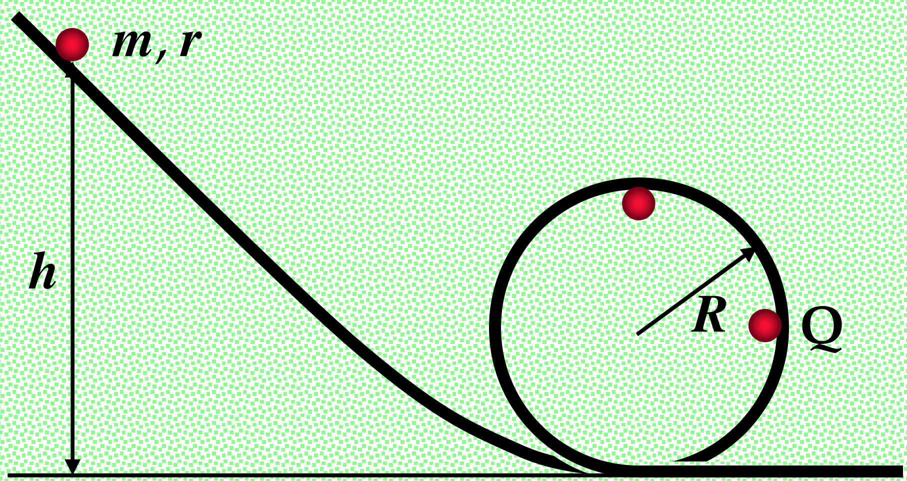
Chapter 13: Rotation of a Rigid Body

example: The "loop-the-loop" revisited…

We just now need to equate $E_{M,i}$ to $E_{M,f}$, and solve for h:

$$E_{M,i} = E_{M,f} \Rightarrow \cancel{m}gh = 2.7\cancel{mg}R \Rightarrow \boxed{h = 2.7R}$$

Variations on a theme:



1. ***What is the normal force exerted by the track on the sphere at point Q?***

2. ***If the sphere is released from h = 4R, what is the normal force exerted by the track on the sphere at the top of the loop?***

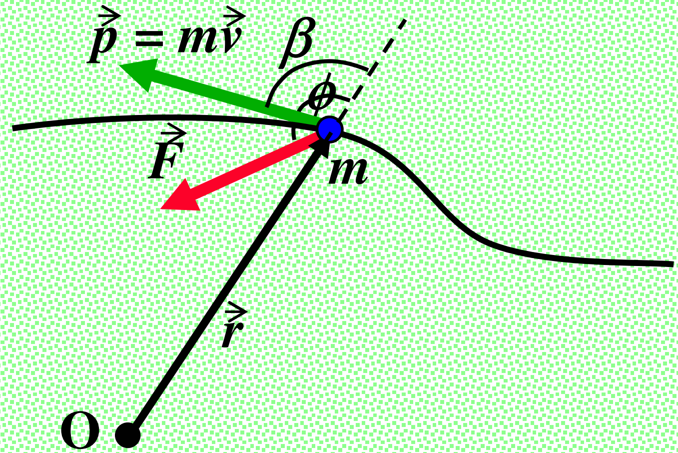
Chapter 13: Rotation of a Rigid Body

Angular momentum (L) (a different approach from your text)

Consider a particle on a trajectory with momentum $\vec{p}$ being watched by an observer at O.

As the particle swings by, the observer *turns her head* in order to keep her eye on it, *even if that particle is moving in a straight line*.

Thus, one ought to be able to describe the motion with *angular variables* as well as linear variables.



Provisional mathematical definition of angular momentum:

$$L = rp\sin\beta = rmv\sin\beta$$

This is entirely analogous to how torque was defined:

$$\tau = rF\sin\phi = rma\sin\phi$$

Chapter 13: Rotation of a Rigid Body

Like torque, angular momentum depends upon:

1. the magnitude of the linear momentum, $\vec{p}$, of the particle;

2. The magnitude of the displacement, $\vec{r}$, between the "origin" (could be a rotation axis but need not be) to the point mass;

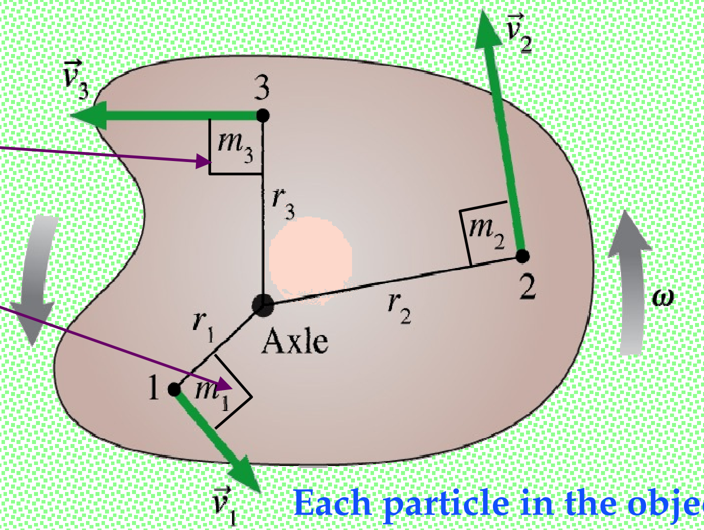
3. The angle between $\vec{r}$ and $\vec{p}$.

Angular momentum of a rigid body about a rotation axis:

$\beta = 90°$ for each point $\Rightarrow$ $\sin\beta = 1$

Every point rotates with the same angular speed, ω. Thus, $v_i = r_i\,\omega$.

$$L_{tot} = \sum_{i=1}^{N} r_i m_i v_i = \underbrace{\sum_{i=1}^{N} m_i r_i^2}_{I}\,\omega = I\omega$$



Each particle in the object has linear momentum as the object rotates.

Link between angular momentum and torque:

1. For solid body rotation:

$$\frac{dL}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} = I\alpha = \tau_{\text{net}}$$

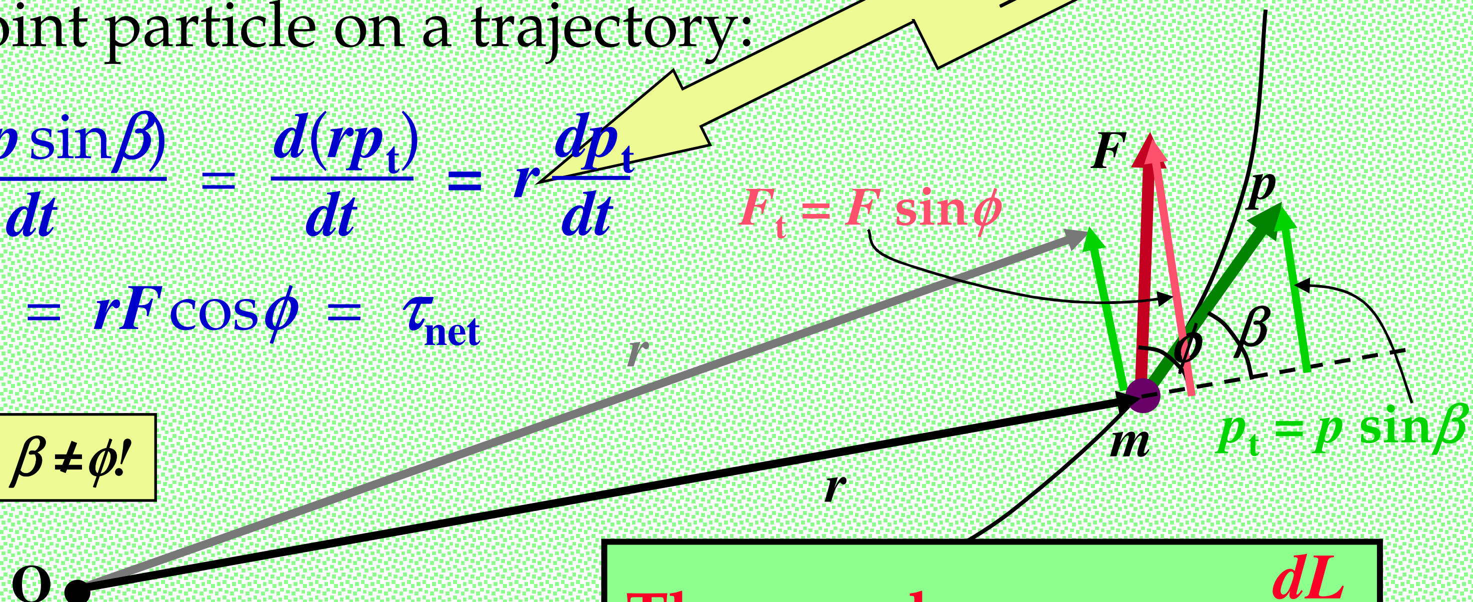
tangential momentum
⇒ *r* is constant

2. For a point particle on a trajectory:

$$\frac{dL}{dt} = \frac{d(rp\sin\beta)}{dt} = \frac{d(rp_{\text{t}})}{dt} = r\frac{dp_{\text{t}}}{dt}$$

$$= rF_{\text{t}} = rF\cos\phi = \tau_{\text{net}}$$

In general, $\beta \neq \phi$!



Thus, we have: $\tau_{\text{net}} = \frac{dL}{dt}$

Chapter 13: Rotation of a Rigid Body

13.9 The vector description of rotational motion

Angular velocity: $\vec{\omega}$ = (ω, direction given by right hand rule)

Angular acceleration: $\vec{\alpha}$ = (α, direction given by R H rule)

Rotation confined to the *x-y* plane:

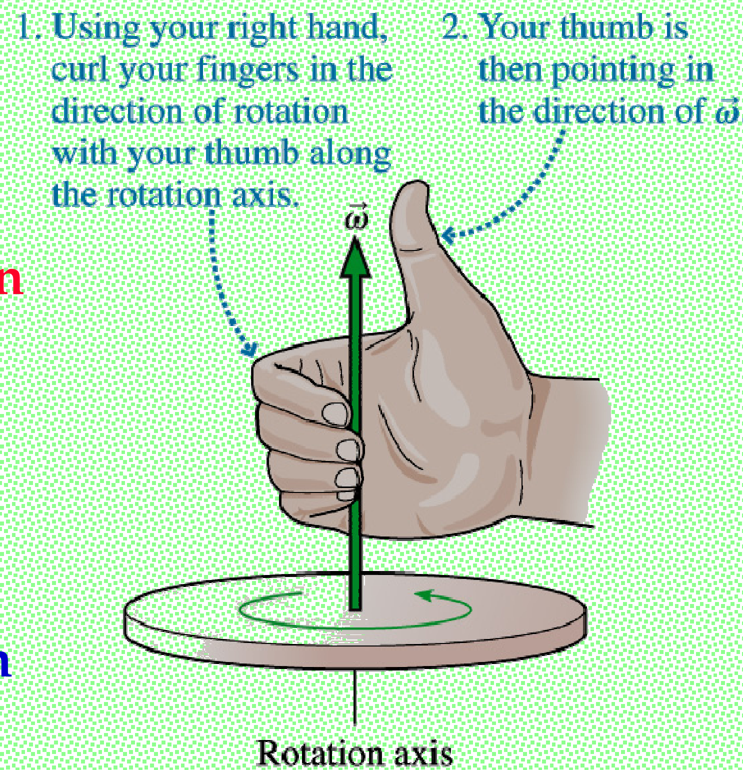
ccw rotation $\Rightarrow$ +*z* direction

cw rotation $\Rightarrow$ –*z* direction

Why should ω and α point in the *z*-direction if the motion is in the *x-y* plane?

What direction in the *x-y* plane would one choose? The direction of motion keeps changing there!

The *z*-direction is the only unique direction implicated by rotation in the *x-y* plane.



Chapter 13: Rotation of a Rigid Body

The vector description of rotational motion, continued…

Angular momentum: $\vec{L}$ = (L, direction given by R H rule)

Torque: $\vec{\tau}$ = (τ, direction given by right hand rule)

Mechanics table

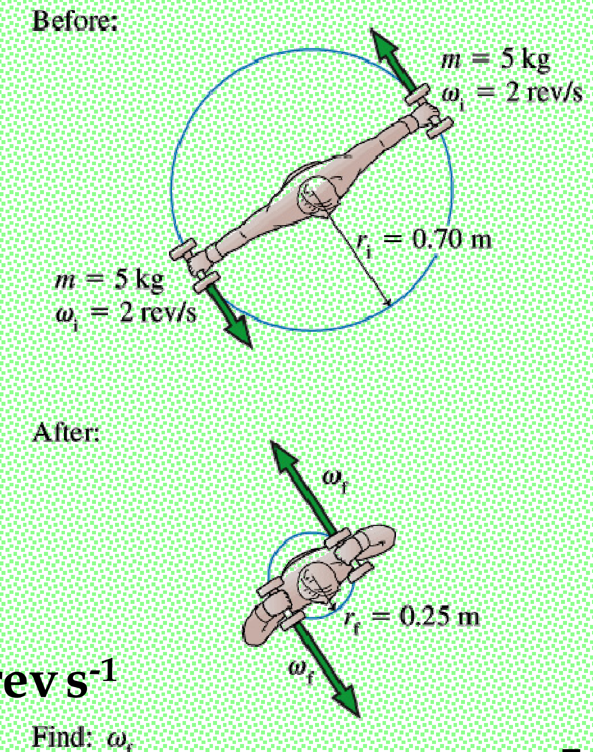
links	linear mechanics	angular mechanics
$v_t = \omega r;\ a_t = \alpha r$	$\Delta x = v_0 \Delta t + \frac{1}{2} a \Delta t^2$	$\Delta \theta = \omega_0 \Delta t + \frac{1}{2} \alpha \Delta t^2$
$\tau = rF \sin\phi$	$\vec{F}_{net} = \frac{d\vec{p}}{dt} = m\vec{a}$	$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} = I\vec{\alpha}$
$I = \sum_{i=1}^{N} m_i r_i^2$	$K_{lin} = \frac{1}{2} m v^2$	$K_{rot} = \frac{1}{2} I \omega^2$
$L = rp \sin\beta$	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{\omega}$

Chapter 13: Rotation of a Rigid Body

Our third conservation law: **Conservation of angular momentum.**

In a system in which there are no external torques, angular momentum is conserved.

$$\text{if} \quad \frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}} = 0 \;\Rightarrow\; \vec{L} = \text{constant}$$



A figure skater, $I_0 = 0.80$ kg m^2 about his central axis, spins at 2 rev s^{-1} holding out two masses, $m = 5$ kg, at $r_i = 0.70$ m. What is his angular speed when he brings the masses to $r_f = 0.25$ m?

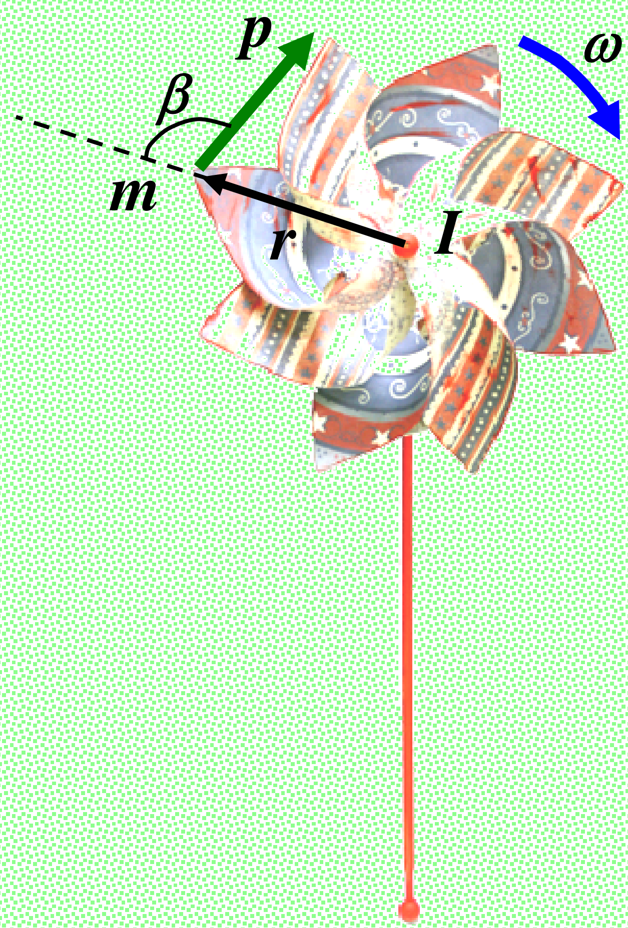
Conserve L: $L_f = L_i \;\Rightarrow\; I_f\,\omega_f = I_i\,\omega_i$

$$\left.\begin{array}{l} I_i = I_0 + 2mr_i^2 = 5.70 \text{ kg m}^2 \\ I_f = I_0 + 2mr_f^2 = 1.43 \text{ kg m}^2 \end{array}\right\} \Rightarrow \omega_f = \frac{I_i}{I_f}\omega_i$$

$$= 4.0\,\omega_i = 8 \text{ rev s}^{-1}$$

Chapter 13: Rotation of a Rigid Body

example: **A pinwheel with moment of inertia about its axis, I_p, and radius r is struck by a mass of chewing gum, m, with momentum p at an angle β as shown. If the gum sticks to the pinwheel, what is its angular speed, ω, immediately after collision?**



Use conservation of angular momentum.

For such problems, it's critical to use the same reference point for both "before" and "after" states.

Since the pinwheel spins about its axis in the "after" state, we use the axis as the reference point for both states.

$$\left.\begin{aligned} L_i &= rp\sin\beta \\ L_f &= I\omega \\ I &= I_p + mr^2 \end{aligned}\right\} \quad L_f = L_i \;\Rightarrow\; I\omega = rp\sin\beta$$

$$\Rightarrow \quad \omega = \frac{rp\sin\beta}{I_p + mr^2}$$

Chapter 13: Rotation of a Rigid Body

This is my very favourite problem! (But don't worry, you won't see it on the exam!)

To put "top English" on a billiard ball, you strike it sharply with a level cue near the top of the ball, as in the diagram. If the ball is struck at height $h = 4r/5$ above its centre and given an initial speed v_0, what is its speed when it stops slipping?

First response: Are you kidding me?

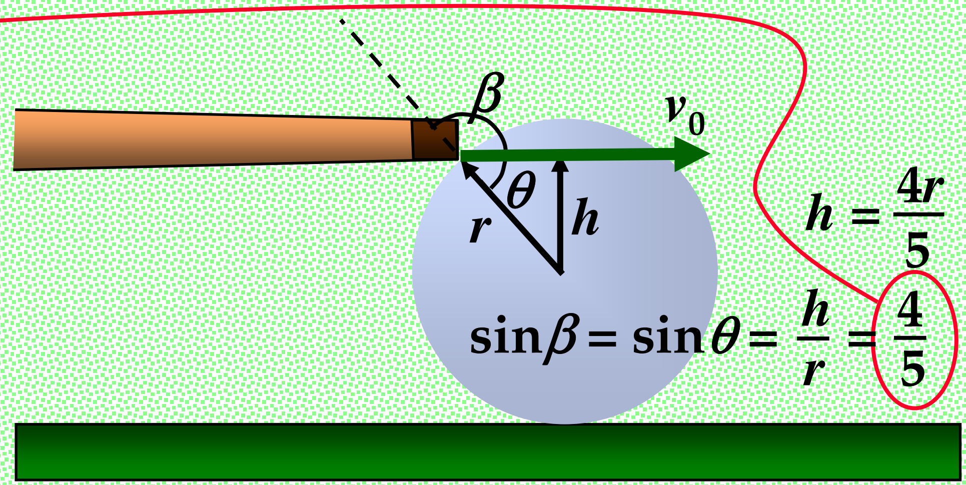
Cue imparts an impulse $J = \Delta p = mv_0$ to the cue ball.

$$\Rightarrow \quad \Delta L = -r\Delta p \sin\beta = -\tfrac{4}{5}mv_0 r$$

$$= I\omega_0 = \tfrac{2}{5}mr^2\omega_0$$

$$\Rightarrow \quad \omega_0 = -\frac{2v_0}{r} \quad (1)$$

negative sign because ω_0 is cw

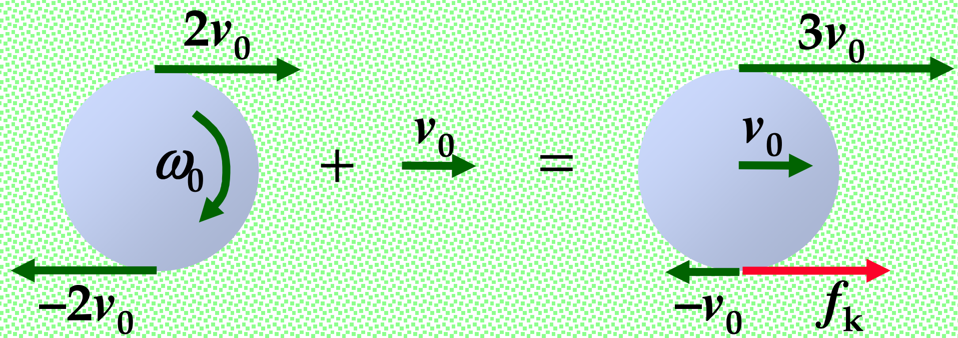


Chapter 13: Rotation of a Rigid Body

greatest physics problem ever, continued…

$$\omega_0 = -\frac{2v_0}{r}$$

$\Rightarrow$ ball's surface speed is $2v_0$ in ball's reference frame…



add centre of mass velocity v_0…

and bottom of ball moves *backwards* at $-v_0$ just after impact $\Rightarrow$ the kinetic friction points right!

$f_k = ma > 0 \Rightarrow$ ball speeds up!

$\tau_k = rf_k = I\alpha > 0 \Rightarrow$ slows negative angular velocity.

$$a = \frac{f_k}{m} \qquad \alpha = \frac{rf_k}{I}$$

Kinematics

$$v_f = v_0 + at = v_0 + \frac{f_k t}{m}$$

$$\Rightarrow f_k t = m(v_f - v_0) \qquad (2)$$

$$\omega_f = \omega_0 + \alpha t = -\frac{2v_0}{r} + \frac{rf_k t}{I} \qquad (3)$$

Chapter 13: Rotation of a Rigid Body

greatest physics problem ever, continued…

Condition for when final velocity is reached: when no-slip conditions are established which ceases f_k.

$$\omega_f = -\frac{v_f}{r} \quad (\text{cw} \Rightarrow \text{negative})$$

set this to (3), and substitute (2) for $f_k t$:

$$\omega_f = -\frac{2v_0}{r} + \frac{r\,m(v_f - v_0)}{\frac{2}{5}mr^2} = -\frac{v_f}{r}$$

$$\Rightarrow \quad -2v_0 + \frac{5}{2}v_f - \frac{5}{2}v_0 = -v_f$$

$$\Rightarrow \quad \frac{7}{2}v_f = \frac{9}{2}v_0 \quad \Rightarrow \quad \boxed{v_f = \frac{9}{7}v_0}$$

from last slide…

$$f_k t = m(v_f - v_0) \qquad (2)$$

$$\omega_f = -\frac{2v_0}{r} + \frac{r f_k t}{I} \qquad (3)$$

and that's all she wrote!

☺ ☺ ☺

ended here, 27/11/08

Chapter 13: Rotation of a Rigid Body

The remaining slides define angular mechanics in terms of vectors, including the cross product. This material will not be on the final exam, and is included here only for your information.

The only real thing you should know here is that our "provisional" definitions of torque and angular momentum are fine for single particles and uniform rigid bodies of suitable symmetry. The general problem of rotational dynamics requires intimate knowledge of vector and indeed tensor analysis, much of which is beyond the scope of a first year course in physics.

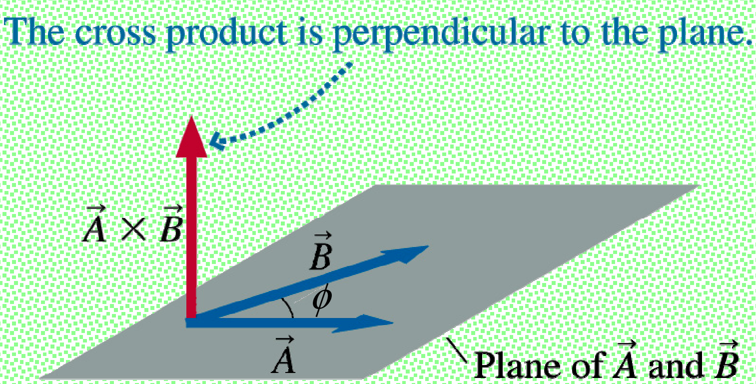
Chapter 13: Rotation of a Rigid Body

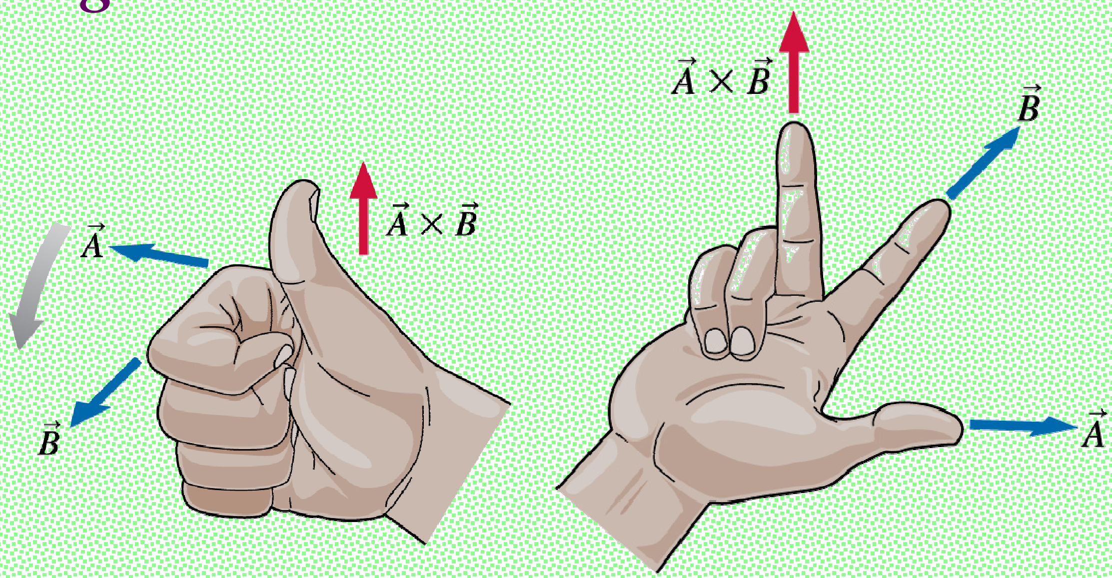
The "cross product"

The cross (vector) product multiplies two vectors together and gives another vector. Let $\vec{A}$ and $\vec{B}$ be two vectors. Then

$$\vec{A} \times \vec{B} = (AB\sin\phi, \text{ direction given by the } \textbf{\textit{right hand rule}})$$

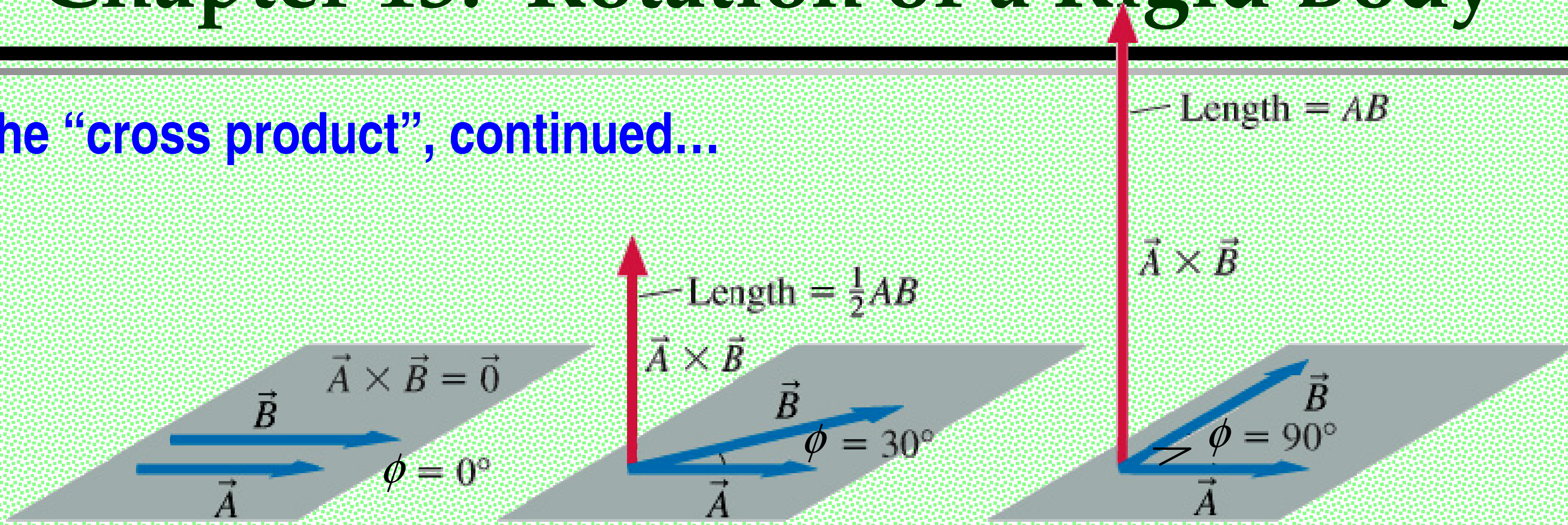
where ϕ is the **smaller** angle (less than 180°) between $\vec{A}$ and $\vec{B}$ *positioned with their tails together*.





Chapter 13: Rotation of a Rigid Body

The “cross product”, continued…



The cross product is zero when $\vec{A}$ and $\vec{B}$ are parallel.

As ϕ increases from 0° to 90°, the length of $\vec{A} \times \vec{B}$ increases.

The cross product is maximum when $\vec{A}$ and $\vec{B}$ are perpendicular.

Thus, for the unit vectors, $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$, we have:

$$\hat{\imath} \times \hat{\jmath} = \hat{k} \qquad \hat{\jmath} \times \hat{\imath} = -\hat{k}$$

$$\hat{\jmath} \times \hat{k} = \hat{\imath} \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath} \qquad \hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$$

$$\hat{k} \times \hat{\imath} = \hat{\jmath} \qquad \hat{\imath} \times \hat{k} = -\hat{\jmath}$$

Chapter 13: Rotation of a Rigid Body

The "cross product", continued…

Note that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (direct result of the right-hand rule)

Calculating the cross product with components:

$$\vec{A} = A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k} \qquad \vec{B} = B_x \hat{\imath} + B_y \hat{\jmath} + B_z \hat{k}$$

$$\begin{aligned}
\Rightarrow \quad \vec{A} \times \vec{B} &= (A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k}) \times (B_x \hat{\imath} + B_y \hat{\jmath} + B_z \hat{k}) \\
&= A_x B_x \, \hat{\imath} \times \hat{\imath} + A_x B_y \, \hat{\imath} \times \hat{\jmath} + A_x B_z \, \hat{\imath} \times \hat{k} \\
&\quad + A_y B_x \, \hat{\jmath} \times \hat{\imath} + A_y B_y \, \hat{\jmath} \times \hat{\jmath} + A_y B_z \, \hat{\jmath} \times \hat{k} \\
&\quad + A_z B_x \, \hat{k} \times \hat{\imath} + A_z B_y \, \hat{k} \times \hat{\jmath} + A_z B_z \, \hat{k} \times \hat{k}
\end{aligned}$$

Chapter 13: Rotation of a Rigid Body

The "cross product", continued…

Note that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (direct result of the right-hand rule)

Calculating the cross product with components:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \qquad \vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

$$\Rightarrow \quad \vec{A} \times \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$

$$= A_xB_x\,\hat{\imath}\times\hat{\imath}\,(0) + A_xB_y\,\hat{\imath}\times\hat{\jmath} + A_xB_z\,\hat{\imath}\times\hat{k}$$

$$+ A_yB_x\,\hat{\jmath}\times\hat{\imath} + A_yB_y\,\hat{\jmath}\times\hat{\jmath}\,(0) + A_yB_z\,\hat{\jmath}\times\hat{k}$$

$$+ A_zB_x\,\hat{k}\times\hat{\imath} + A_zB_y\,\hat{k}\times\hat{\jmath} + A_zB_z\,\hat{k}\times\hat{k}\,(0)$$

Chapter 13: Rotation of a Rigid Body

The "cross product", continued…

Note that $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ (direct result of the right-hand rule)

Calculating the cross product with components:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \qquad \vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

$$\Rightarrow \quad \vec{A}\times\vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})\times(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$

$$= A_xB_x(0) + A_xB_y(\hat{k}) + A_xB_z(-\hat{\jmath})$$

$$+ A_yB_x(-\hat{k}) + A_yB_y(0) + A_yB_z(\hat{\imath})$$

$$+ A_zB_x(\hat{\jmath}) + A_zB_y(-\hat{\imath}) + A_zB_z(0)$$

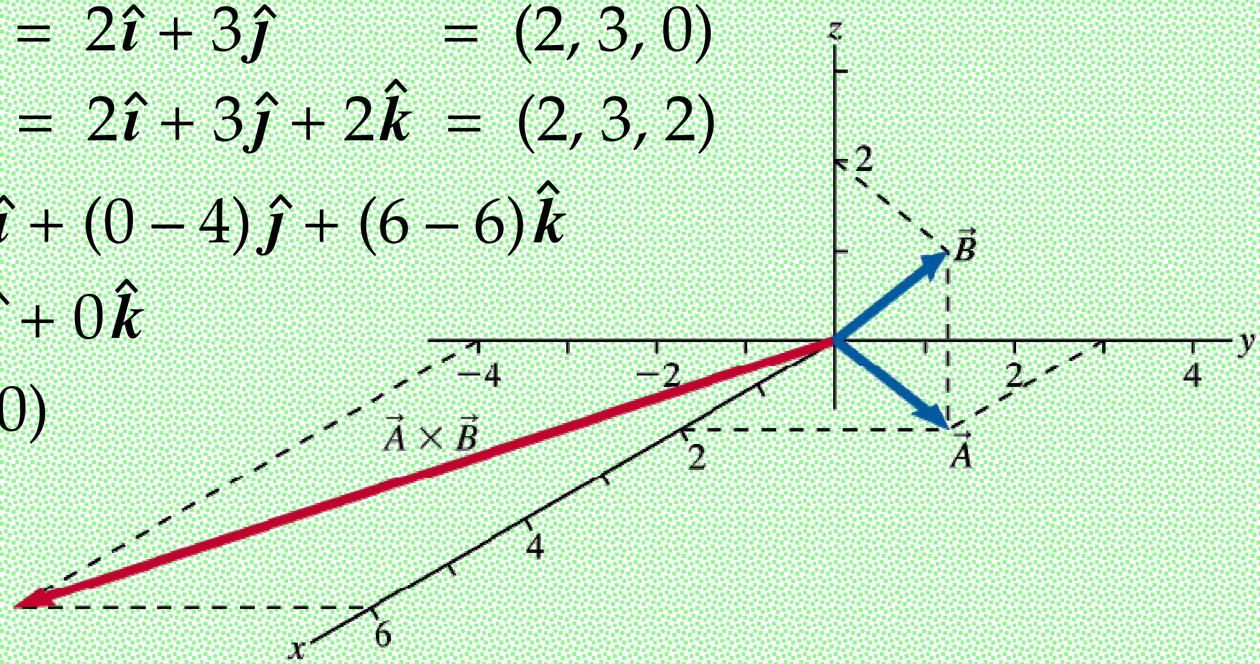
$$\Rightarrow \quad \vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}$$

Chapter 13: Rotation of a Rigid Body

quick example:

$$\vec{A} = 2\hat{\imath} + 3\hat{\jmath} = (2, 3, 0)$$
$$\vec{B} = 2\hat{\imath} + 3\hat{\jmath} + 2\hat{k} = (2, 3, 2)$$

$$\Rightarrow \quad \vec{A}\times\vec{B} = (6-0)\hat{\imath} + (0-4)\hat{\jmath} + (6-6)\hat{k}$$
$$= 6\hat{\imath} - 4\hat{\jmath} + 0\hat{k}$$
$$= (6, -4, 0)$$



Differentiation of cross products follows the product rule:

$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$$

Chapter 13: Rotation of a Rigid Body

13.9 The vector description of rotational motion

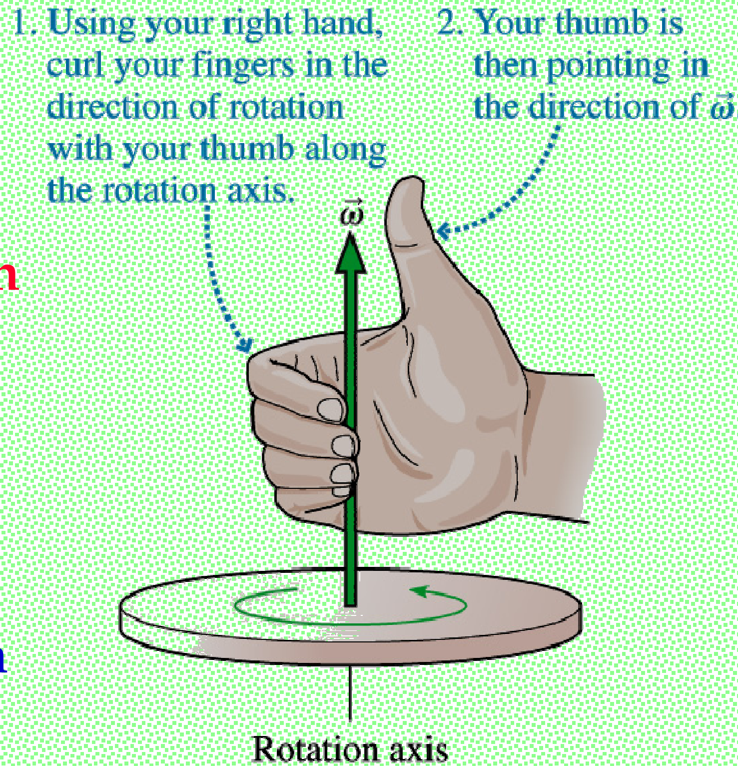
Angular velocity: $\vec{\omega}$ = (ω, direction given by right hand rule)

Angular acceleration: $\vec{\alpha}$ = (α, direction given by R H rule)

Rotation confined to the *x-y* plane:

ccw rotation $\Rightarrow$ $+z$ direction

cw rotation $\Rightarrow$ $-z$ direction



Why should ω and α point in the *z*-direction if the motion is in the *x-y* plane?

What direction in the *x-y* plane would one choose? The direction of motion keeps changing there!

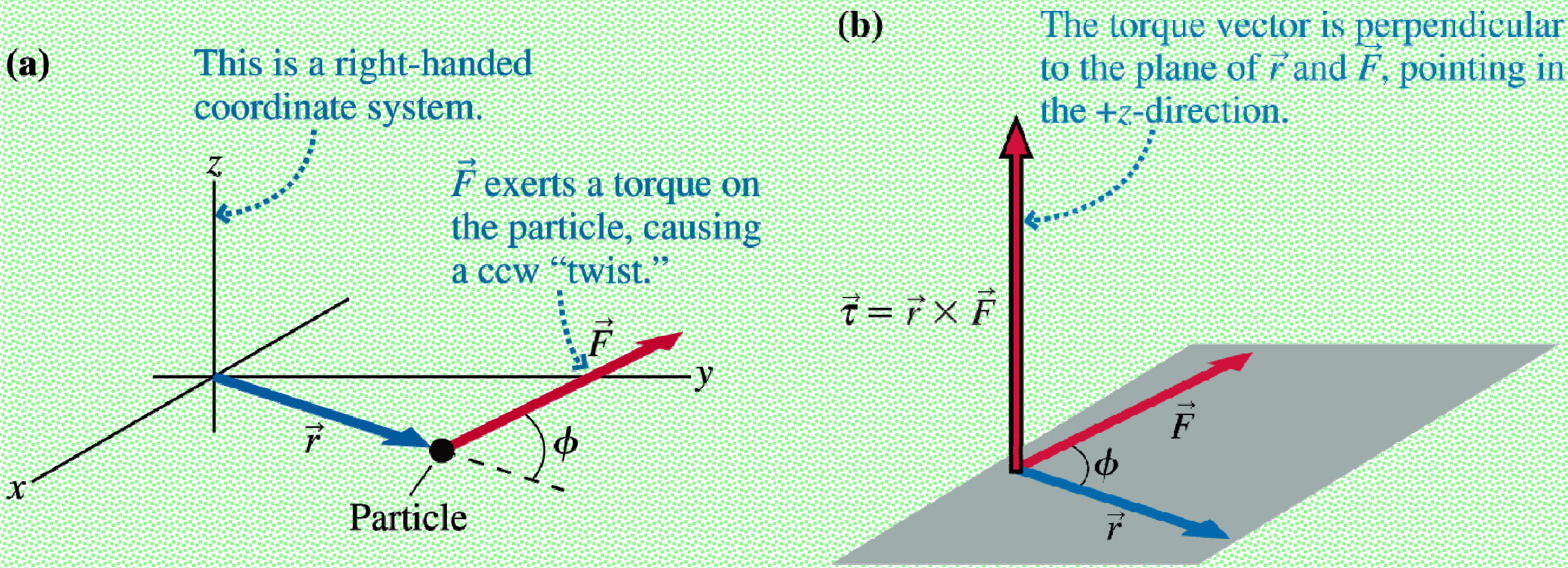
The *z*-direction is the only unique direction implicated by rotation in the *x-y* plane.

Chapter 13: Rotation of a Rigid Body

The vector description of Torque

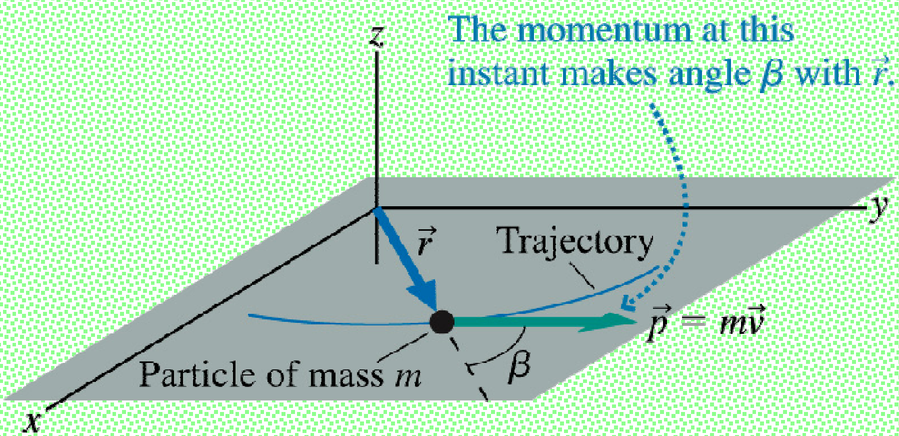
Torque: $\vec{\tau} = \vec{r} \times \vec{F} = (rF\sin\phi$, direction given by R H rule)

Caution: need to move the tails of the vectors together before using right hand rule!

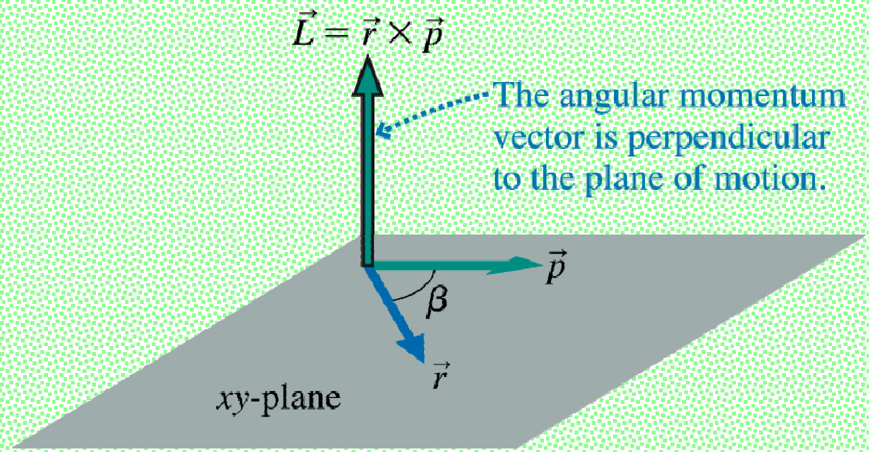


Angular momentum of a particle

$$\vec{L} = \vec{r} \times \vec{p} = (rmv\sin\beta, \text{ direction given by R H rule})$$



The particle is moving along a trajectory.



The vector tails are placed together to determine the cross product.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \underbrace{\vec{v} \times \vec{p}}_{0} + \vec{r} \times \vec{F} = \vec{\tau}$$

$\vec{v} \parallel \vec{p}$

$$\Rightarrow \boxed{\frac{d\vec{L}}{dt} = \vec{\tau}} \quad \text{compare with} \quad \frac{d\vec{p}}{dt} = \vec{F}$$