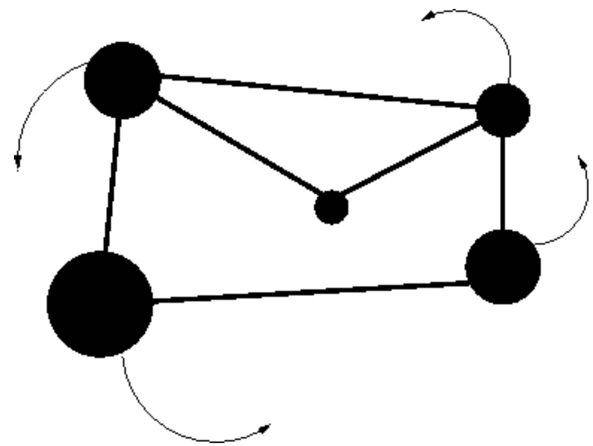
Chapter 9: Rotation of Rigid Bodies

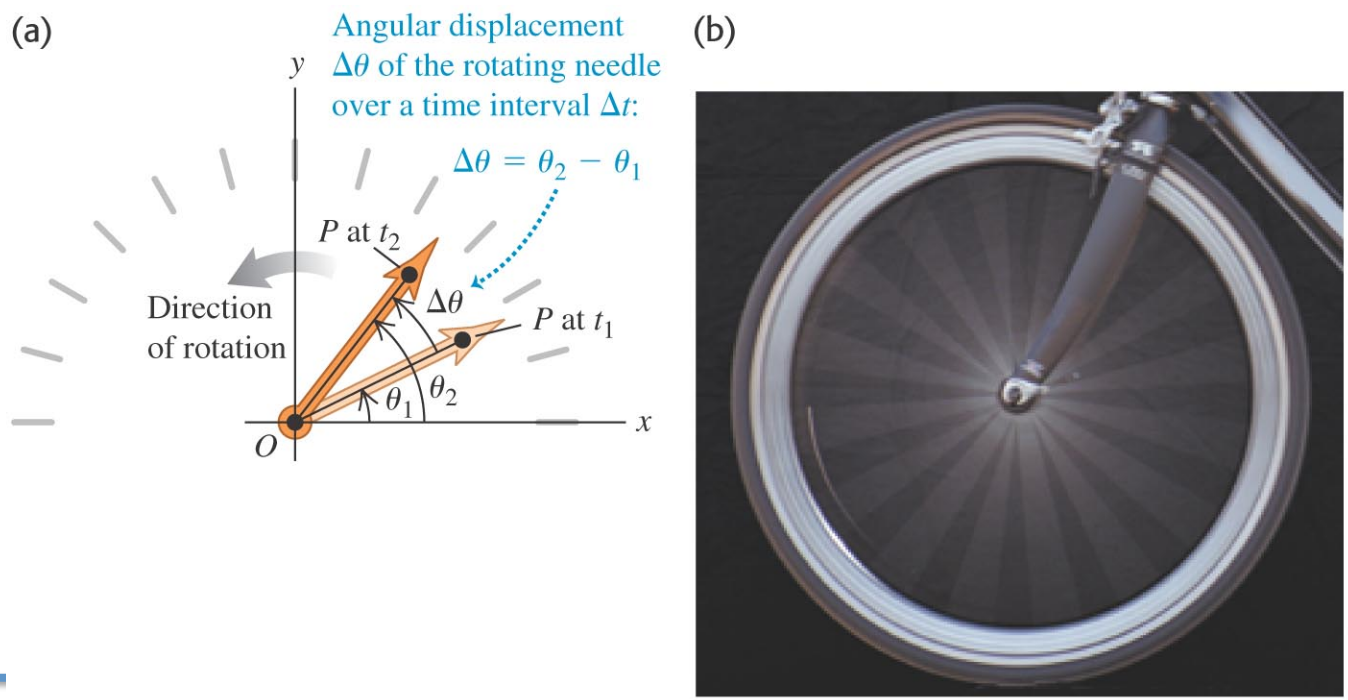
- Rigid rotation means that relative orientation of collection of objects remains constant while object rotates and/or moves.



- This chapter covers the topics of rotational kinematics, rotational kinetic energy, and moment of inertia.

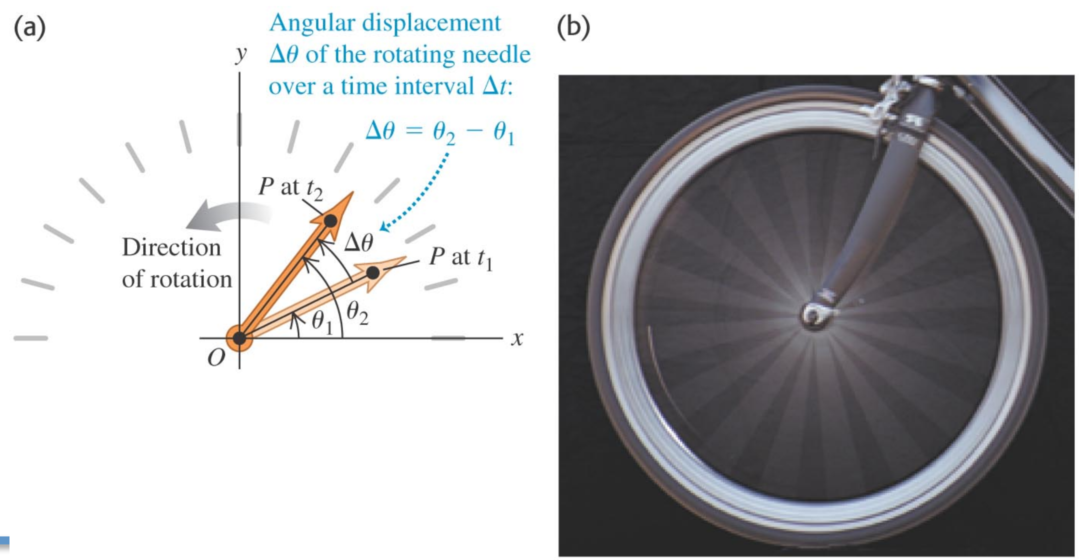
Angular Displacement

- Consider a vector rotating around a point.
- We can define the angle θ to be the angle between this vector and the x axis.
- After a certain time interval, the change in the angle, or the angular displacement, is Δθ.



Angular velocity

- The angular velocity is the rate of change of the angular displacement: ω = dθ/dt (units: radians per second).
  - Note that the angle must be expressed in radians
- Angular frequency is related to frequency: $\omega = 2\pi f$

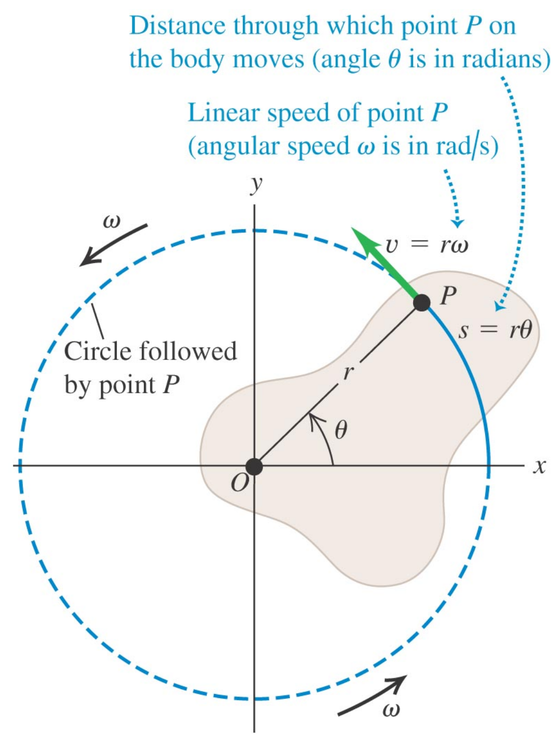


Velocity vs. Angular Velocity

- For a rotating object, the speed of the perimeter is related to the angular velocity:
- Arclength: $s = r\theta$

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

$$v = r\omega$$



(b)

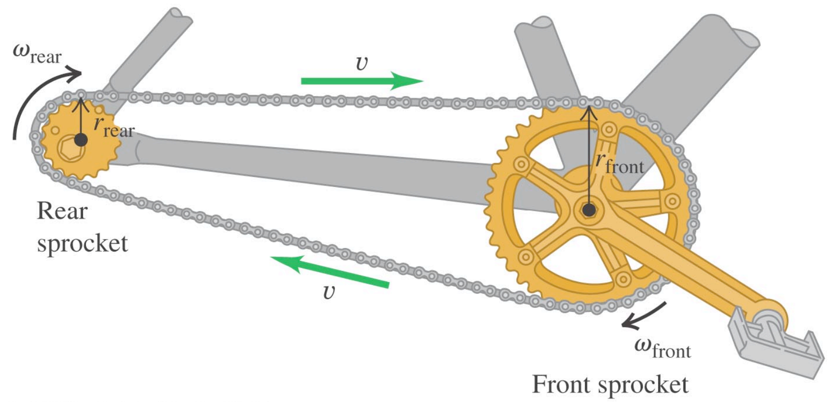
- Alternate derivation:
  - For one cycle, perimeter rotates a distance = circumference

$$v = 2\pi Rf = R\omega$$

Clicker Question

Q9.4

Compared to a gear tooth on the rear sprocket (on the left, of small radius) of a bicycle, a gear tooth on the *front* sprocket (on the right, of large radius) has



A. a faster linear speed and a faster angular speed

B. the same linear speed and a faster angular speed

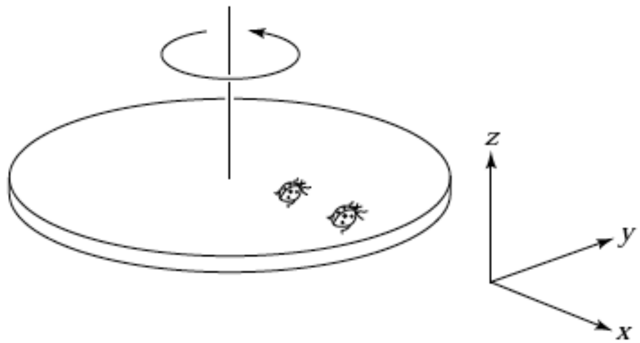
C. a slower linear speed and the same angular speed

D. the same linear speed and a slower angular speed

E. none of the above

Clicker Question

A ladybug sits at the outer edge of a merry-go-round, and a gentleman bug sits halfway between her and the axis of rotation. The merry-go-round makes a complete revolution once each second. The gentleman bug's angular speed is



1. half the ladybug's.
2. the same as the ladybug's.
3. twice the ladybug's.
4. impossible to determine

Pi

Angular Acceleration

- Angular acceleration is the rate of change of the angular velocity

$$\alpha = \frac{d\omega}{dt}$$

- Just as with 1-D kinematics, the angular velocity as a function of time can be determined by integrating the angular acceleration over time:

$$\omega(t) = \omega_0 + \int_0^t \alpha(t)\ dt$$

$$\theta(t) = \theta_0 + \int_0^t \omega(t)\ dt$$

Table 9.1 Comparison of Linear and Angular Motion with Constant Acceleration

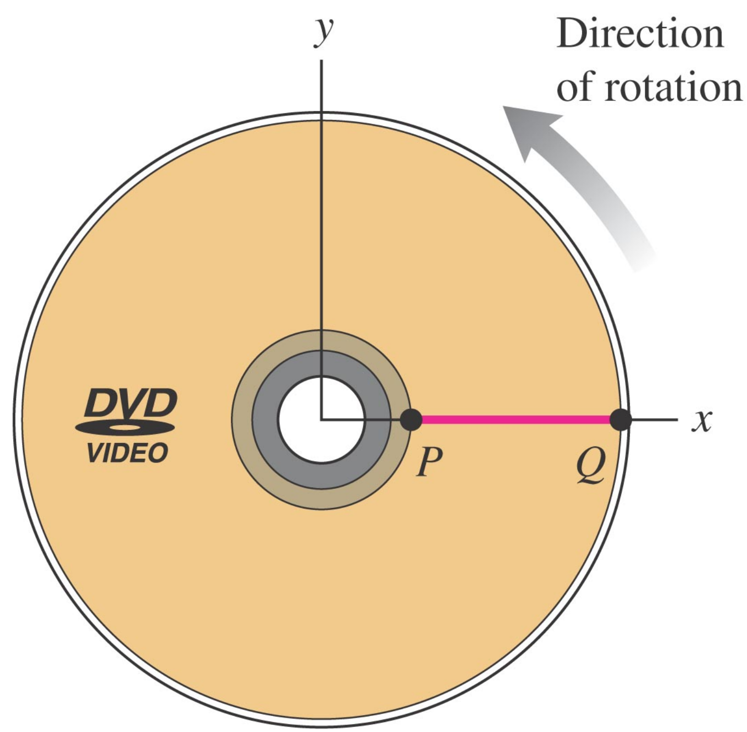
Straight-Line Motion with Constant Linear Acceleration	Fixed-Axis Rotation with Constant Angular Acceleration
$a_x = \text{constant}$	$\alpha_z = \text{constant}$
$v_x = v_{0x} + a_x t$	$\omega_z = \omega_{0z} + \alpha_z t$
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_x + v_{0x})t$	$\theta - \theta_0 = \frac{1}{2}(\omega_z + \omega_{0z})t$

Clicker Question

Q9.2

A DVD is initially at rest so that the line *PQ* on the disc's surface is along the +*x*-axis. The disc begins to turn with a constant $a_z = 5.0\ \mathrm{rad/s^2}$.

At $t = 0.40$ s, what is the angle between the line *PQ* and the +*x*-axis?



A. 0.40 rad

B. 0.80 rad

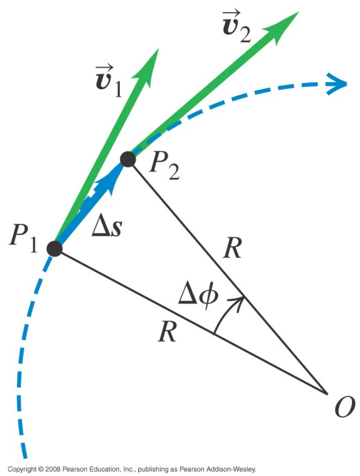
C. 1.0 rad

D. 2.0 rad

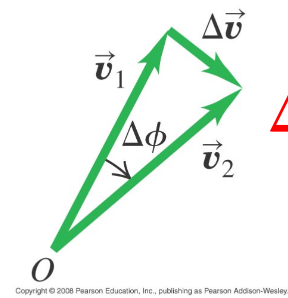
Centripetal (radial) Acceleration

- Recall that an object in circular motion has an acceleration in the (negative) radial direction:

(a) A point moves a distance Δs at constant speed along a circular path.



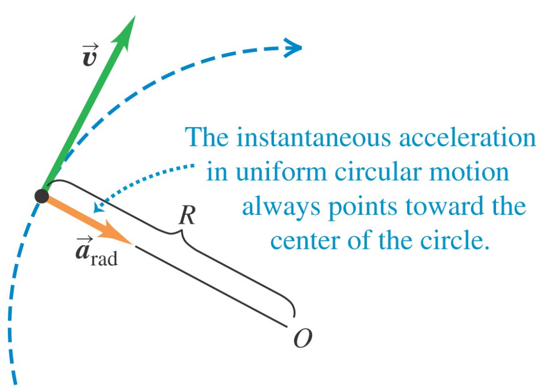
(b) The corresponding change in velocity and average acceleration



$$\Delta v = v\Delta\phi$$

$$a_{cen} = |a_{rad}| = v\frac{d\phi}{dt} = v\omega = \frac{v^2}{R}$$

(c) The instantaneous acceleration



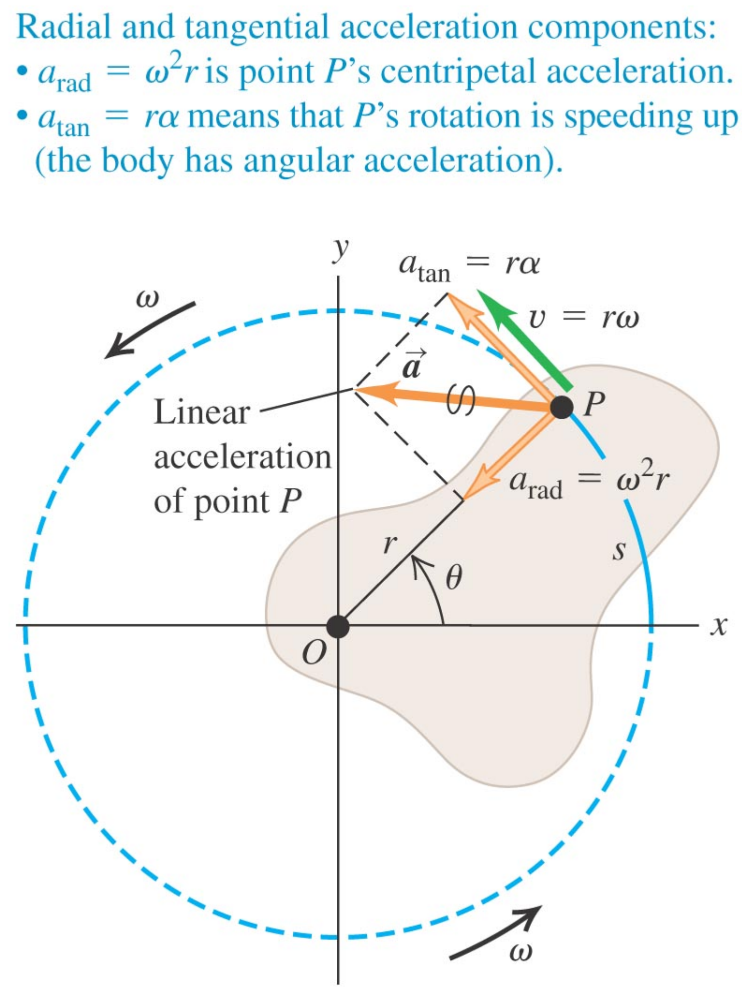
$$\boxed{a_{cen} = \frac{v^2}{R}}$$

Tangential And Radial Acceleration

- If the object in circular motion has an angular acceleration, then the acceleration isn't entirely in the radial direction.
  - The object will have a tangential acceleration (the component of acceleration in the direction of the velocity)

$$a_{tan} = r\alpha$$

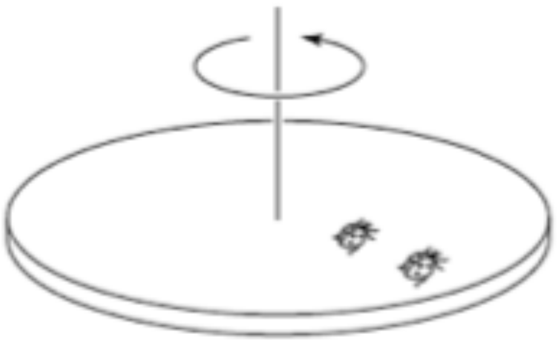
$$\vec{a} = -\frac{v^2}{r}\hat{r} + r\alpha\hat{\theta}$$



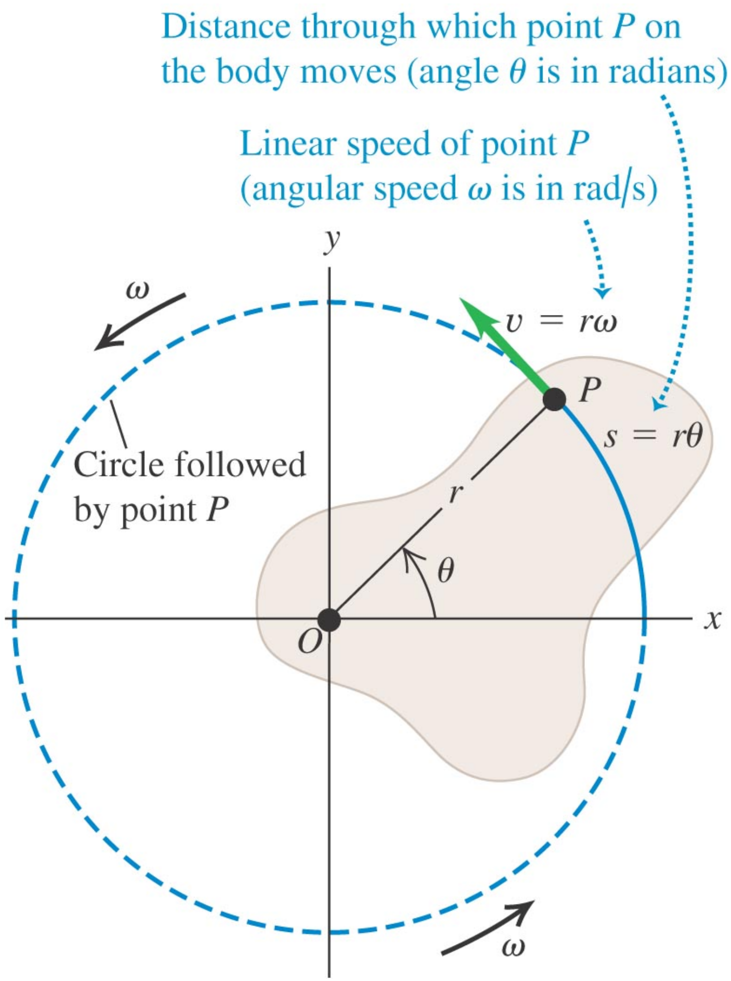
Clicker Question

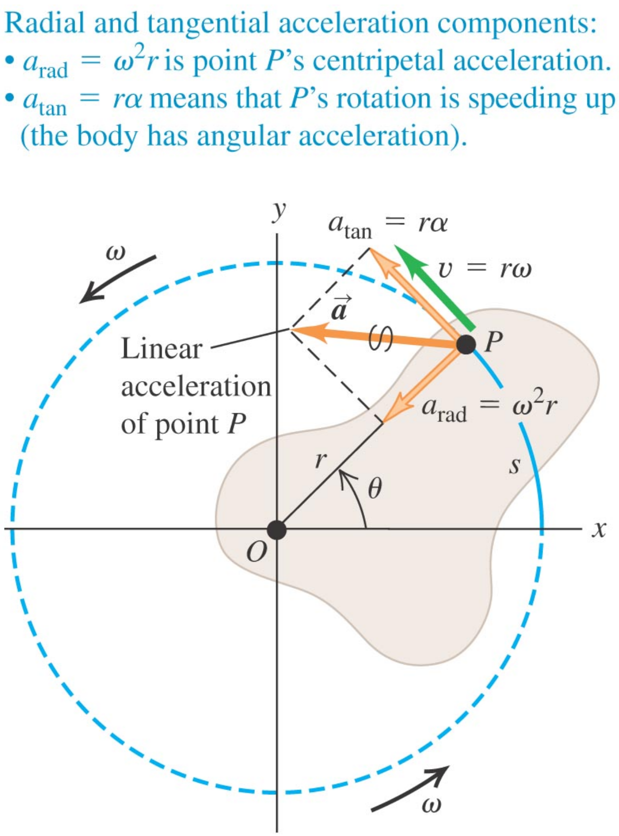
- Recall the two bugs on the merry-go-round. If the angular velocity begins increasing with time (so it rotates faster and faster), which bug is more likely to get flung off first (assume both have the same coefficient of static friction)?

  a. The bug near the edge.
  b. The bug half way between the center and the edge.
  c. Both bugs get flung out at the same time.
  d. Both bugs always stay attached to the merry-go-round, as they have super-duper sticky feet!



Relating Linear and Angular Kinematics





$$\vec{v} = r\omega\hat{\theta}$$

$$\vec{a} = r\omega^2(-\hat{r}) + r\alpha\hat{\theta}$$

Rotational Kinetic Energy and Moment of Inertia

- Consider an object rotating about an axis
- Kinetic energy:

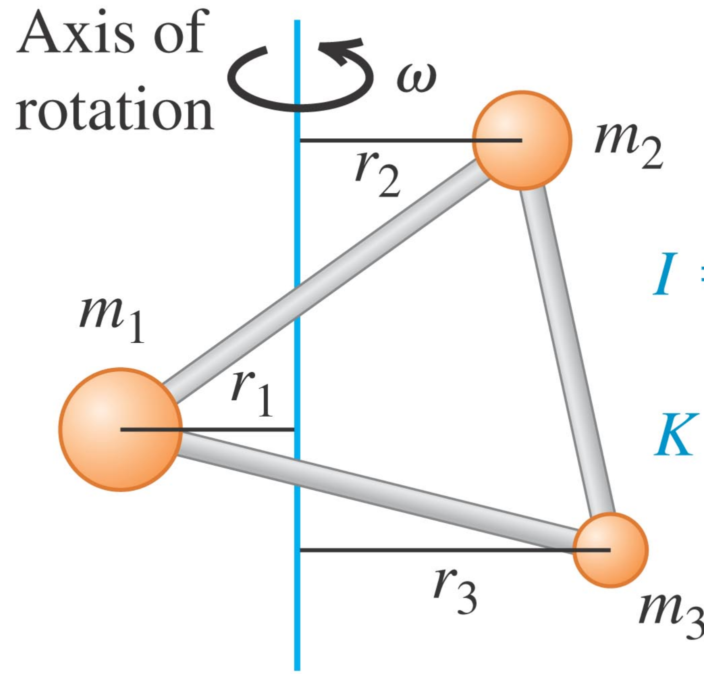
$$K = \sum_i \frac{1}{2} m_i v_i^2$$

$$K = \sum_i \frac{1}{2} m_i (r_i \omega)^2$$

r_i = distance to axis of rotation

$$K = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2$$

$$K = \frac{1}{2} I \omega^2$$



$$I = \sum_i m_i r_i^2$$

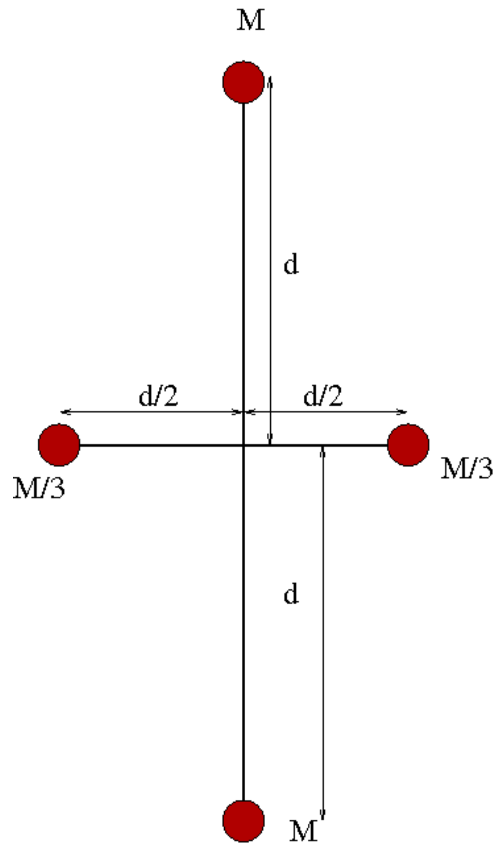
$$K = \frac{1}{2} I \omega^2$$

I is the moment of inertia

$$I = \sum_i m_i r_i^2$$

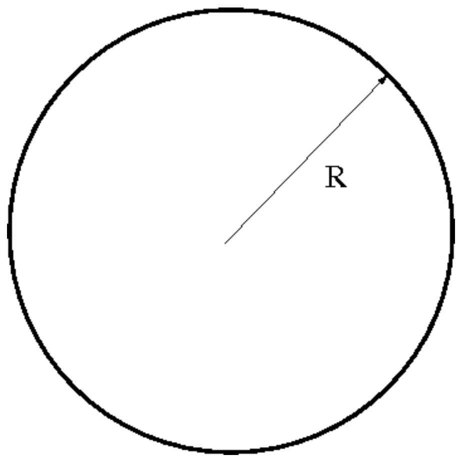
Chalkboard Question

- Determine the moment of inertia of the following object rotating about an axis through the center of mass (perpendicular to the plane of the four masses):



Clicker Question

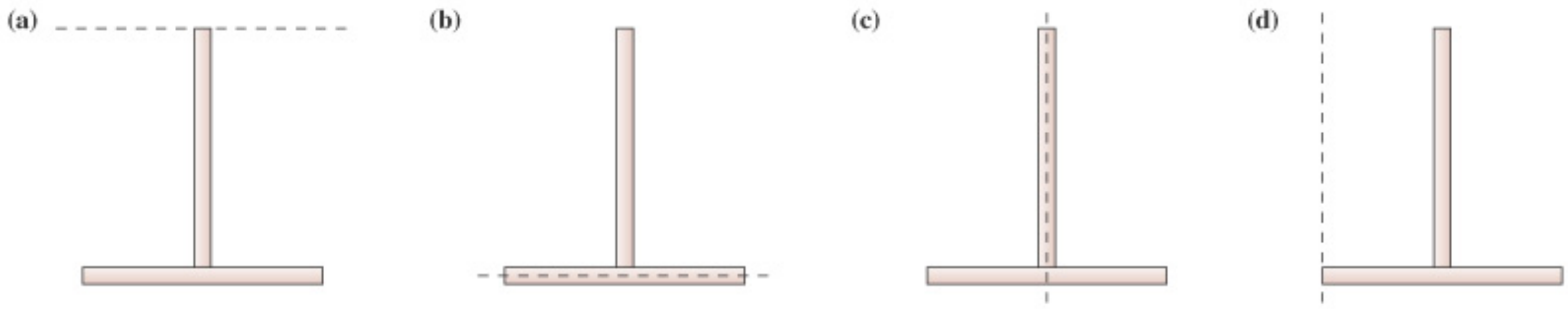
- Determine the moment of inertia of a ring of radius R and mass M (relative to an axis of rotation going through its center perpendicular to the ring)



1. MR^2
2. $0.5\ MR^2$
3. zero
4. 0.5 MR^2 < I < MR^2

Clicker Question

Four Ts are made from two identical rods of equal mass and length. Rank in order, from largest to smallest, the moments of inertia I_a to I_d for rotation about the dotted line.



A. $I_c > I_b > I_d > I_a$
B. $I_c = I_d > I_a = I_b$
C. $I_a = I_b > I_c = I_d$
D. $I_a > I_d > I_b > I_c$
E. $I_a > I_b > I_d > I_c$

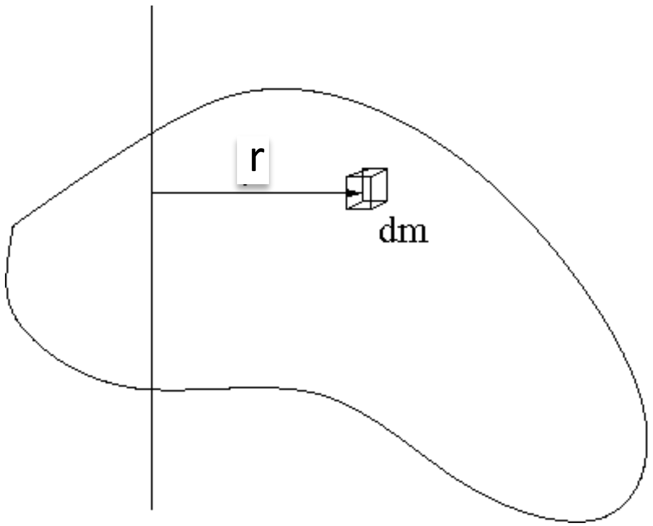
Moment of Inertia for Continuous Mass Distributions

- Recall that the moment of inertia is given by

$$I = \sum_i m_i r_i^2$$

- For a continuous mass distribution, the summation turns into an integral:

$$I = \int r^2 dm$$



  - For 3d distribution:

$$dm = \rho\ dV$$

  - For 2d distributions:

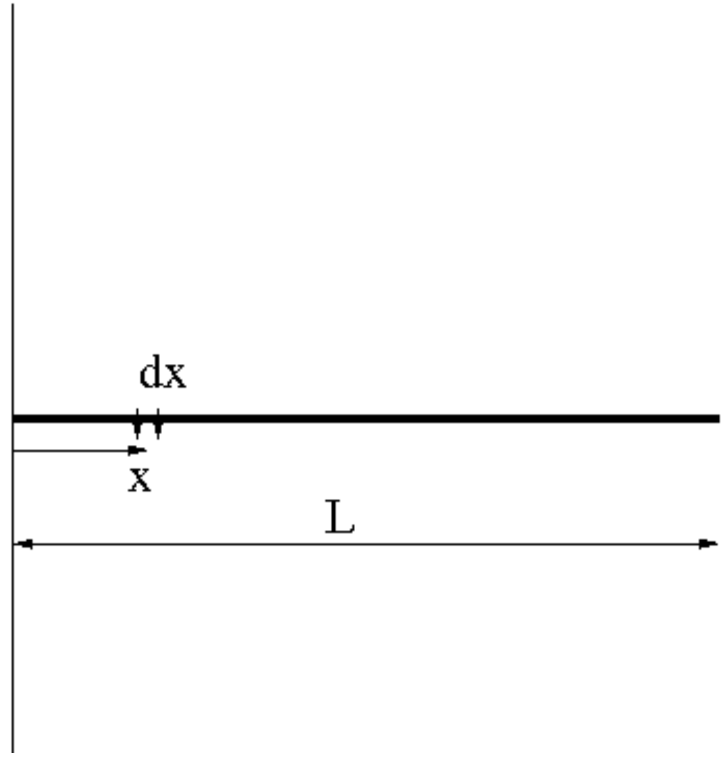
$$dm = \sigma\ dA$$

  - For 1d distributions:

$$dm = \lambda\ dl$$

Chalkboard Question

- Determine the moment of inertia of a uniform rod of length L and mass M, about an axis of rotation through the left end.



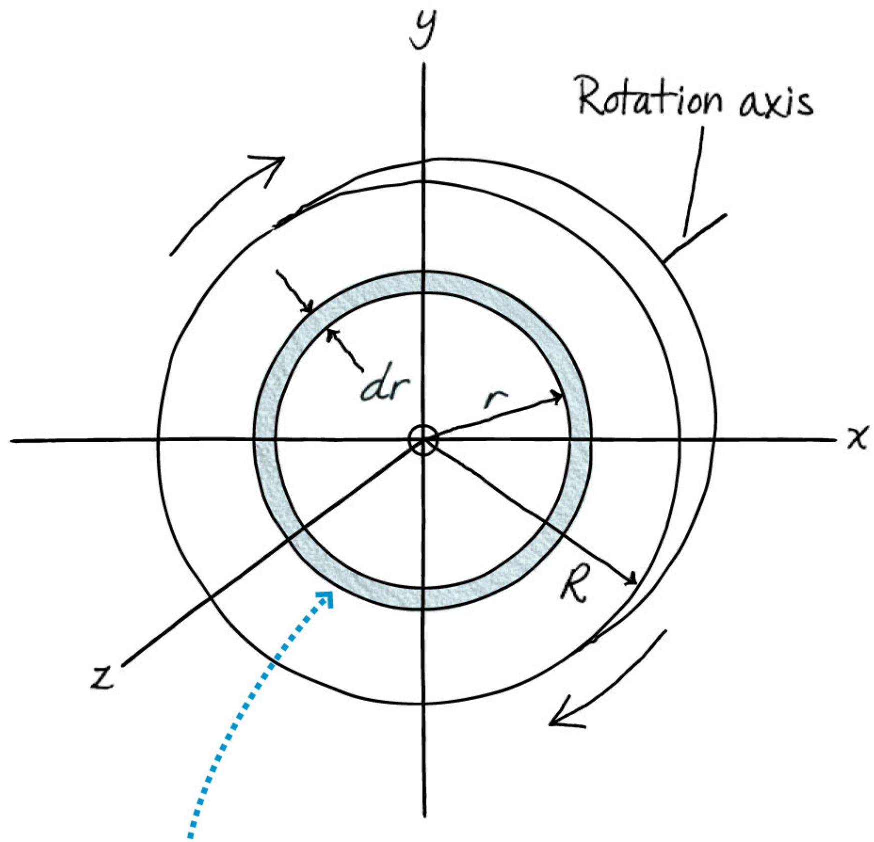
Moment of Inertia of a Solid Cylinder (or Disk)

$$I = \int r^2 dm$$

$$I = \int_0^R r^2 \frac{M}{\pi R^2} 2\pi r \; dr$$

$$I = \frac{1}{4} R^4 \frac{M}{\pi R^2} 2\pi$$

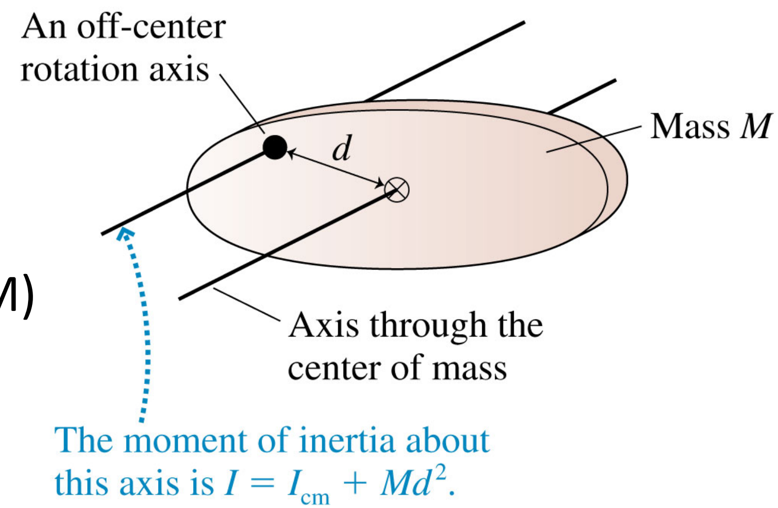
$$I = \frac{1}{2} M R^2$$



A narrow ring of width dr has mass $dm = (M/A)dA$.
Its area is dA = width × circumference = $2\pi r\, dr$.

Parallel Axis Theorem

$$I = \int r^2 dm$$



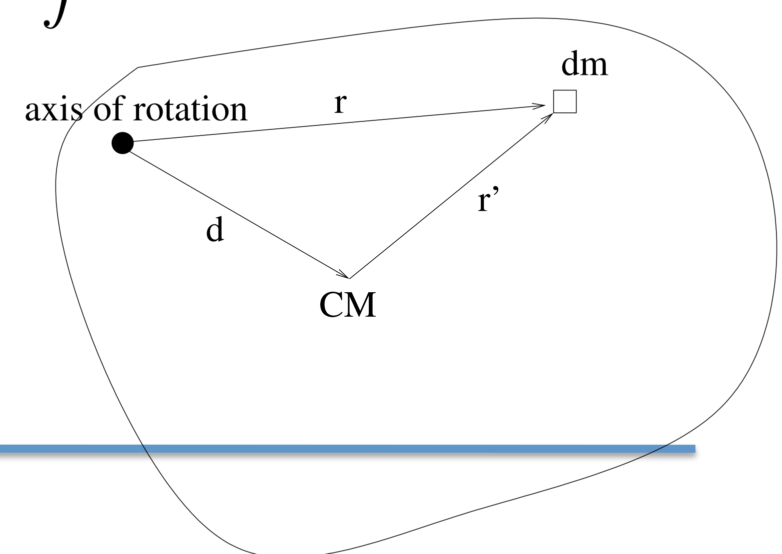
$$\vec{r} = \vec{d} + \vec{r'}$$ ($\vec{r'}$ is with respect to CM)

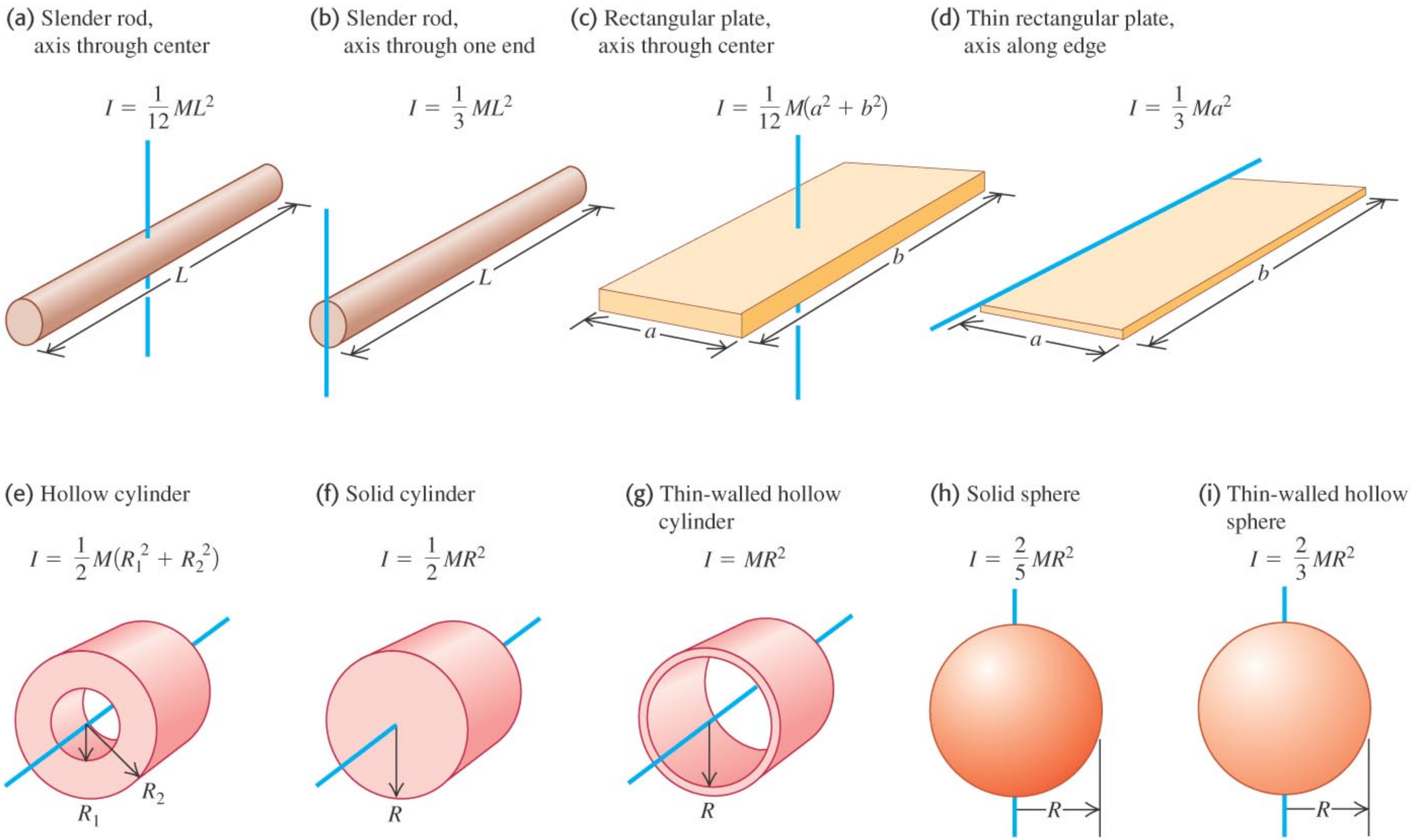
$$r^2 = \vec{r} \cdot \vec{r} = (\vec{d} + \vec{r'}) \cdot (\vec{d} + \vec{r'})$$

$$r^2 = d^2 + (r')^2 + 2\vec{d} \cdot \vec{r'}$$

$$I = \int r^2 \, dm = d^2 M_{tot} + \int (r')^2 \, dm + 2\vec{d} \cdot \int \vec{r'} \, dm$$

$$I = I_{CM} + M_{tot} d^2$$



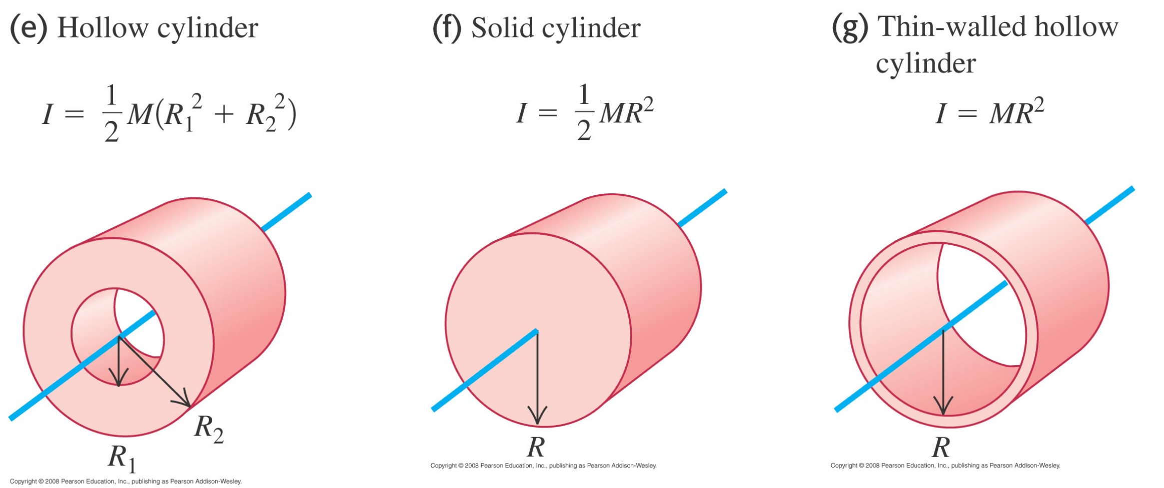


(a) Slender rod, axis through center
$I = \frac{1}{12}ML^2$
L
(b) Slender rod, axis through one end
$I = \frac{1}{3}ML^2$
L
(c) Rectangular plate, axis through center
$I = \frac{1}{12}M(a^2 + b^2)$
a
b
(d) Thin rectangular plate, axis along edge
$I = \frac{1}{3}Ma^2$
a
b
(e) Hollow cylinder
$I = \frac{1}{2}M(R_1^2 + R_2^2)$
R_1
R_2
(f) Solid cylinder
$I = \frac{1}{2}MR^2$
R
(g) Thin-walled hollow cylinder
$I = MR^2$
R
(h) Solid sphere
$I = \frac{2}{5}MR^2$
R
(i) Thin-walled hollow sphere
$I = \frac{2}{3}MR^2$
R

Clicker Question

Q9.6

The three objects shown here all have the same mass M and radius R. Each object is rotating about its axis of symmetry (shown in blue). All three objects have the *same* rotational kinetic energy. Which one is rotating *fastest*?



A. hollow cylinder

B. solid cylinder

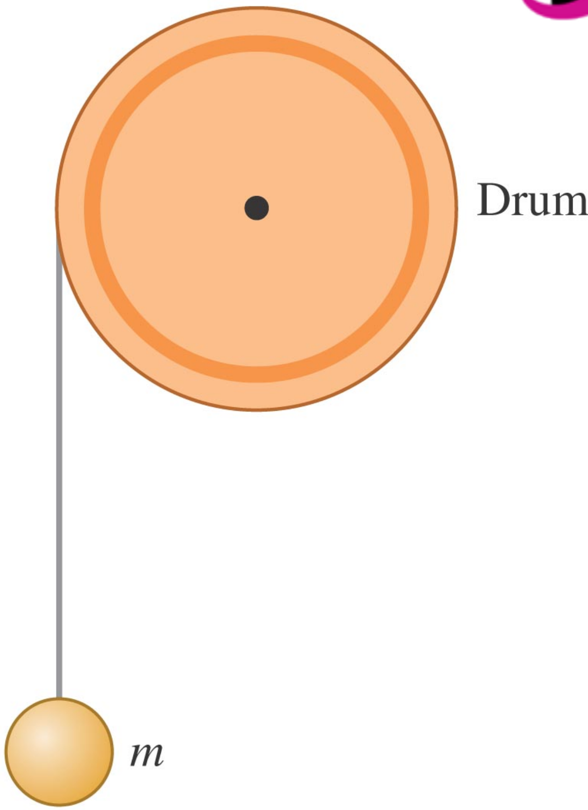
C. thin-walled hollow cylinder

D. two or more of these are tied for fastest

Q9.7

A thin, very light wire is wrapped around a drum that is free to rotate. The free end of the wire is attached to a ball of mass m. The drum has the same mass m. Its radius is R and its moment of inertia is $I = (1/2)mR^2$. As the ball falls, the drum spins.

At an instant that the ball has translational kinetic energy K, the drum has rotational kinetic energy

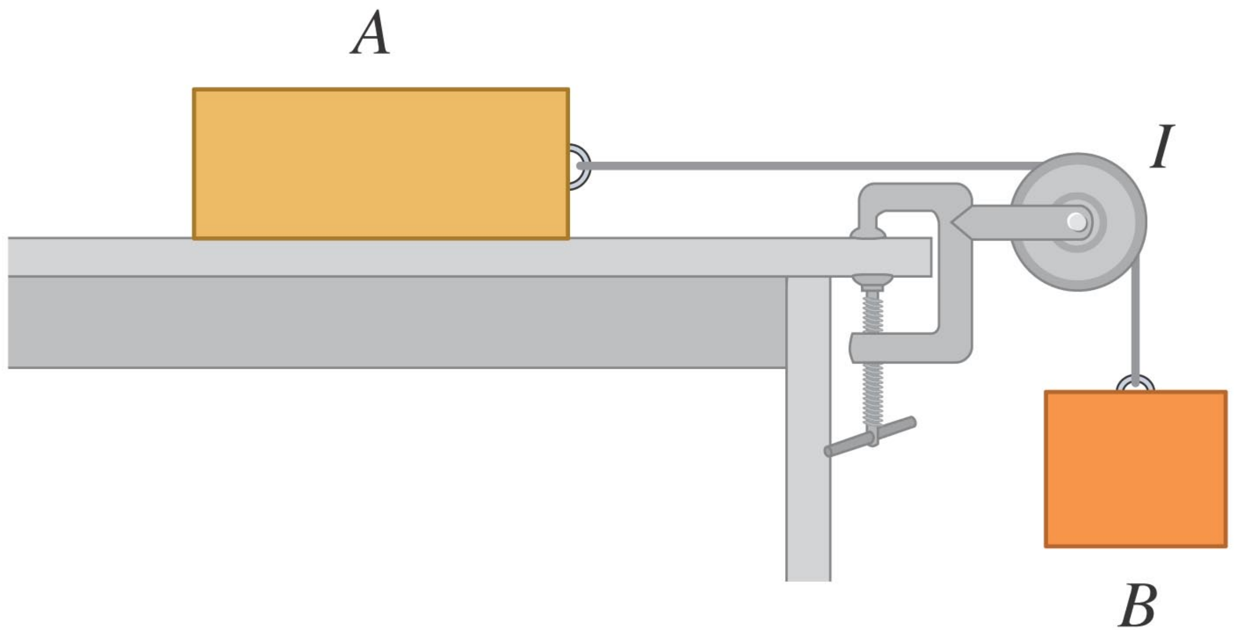


A. K.	B. 2K.	C. K/2.	D. none of these

Using Conservation of Energy Including Rotational Kinetic Energy

$$E = U + K_T + K_R$$

- If Block B falls a distance *h* (starting from rest), what is its final speed (ignore friction; the pulley has a radius R)?



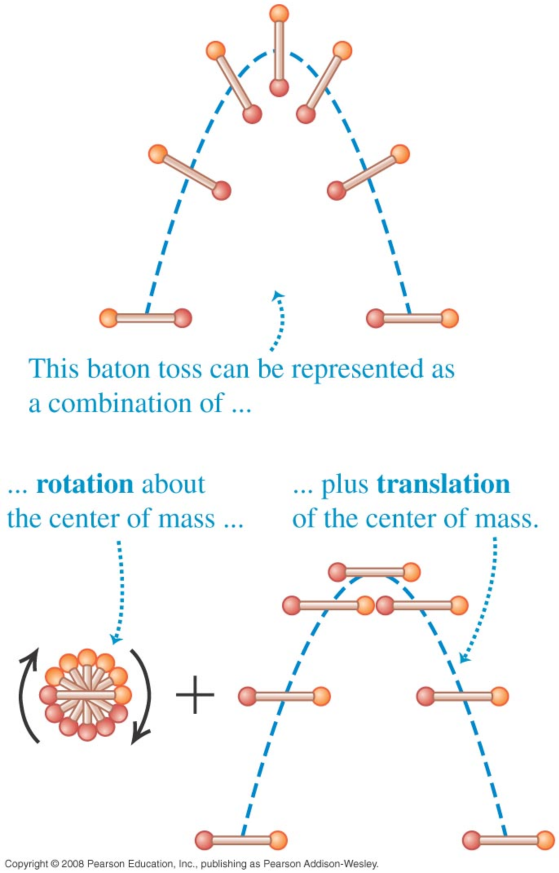
Section 10.3: Rotation About Moving Axis

- All arbitrary motion can be split into pure rotation about the center of mass and pure translational motion
- Kinetic Energy:

$$K = K_T + K_R = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$$

- Rolling without slipping:

$$v_{CM} = R\omega$$



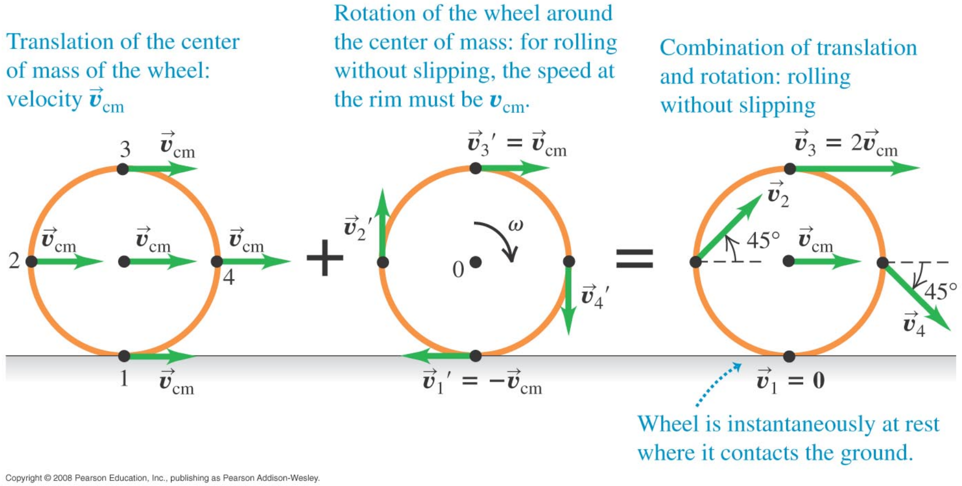
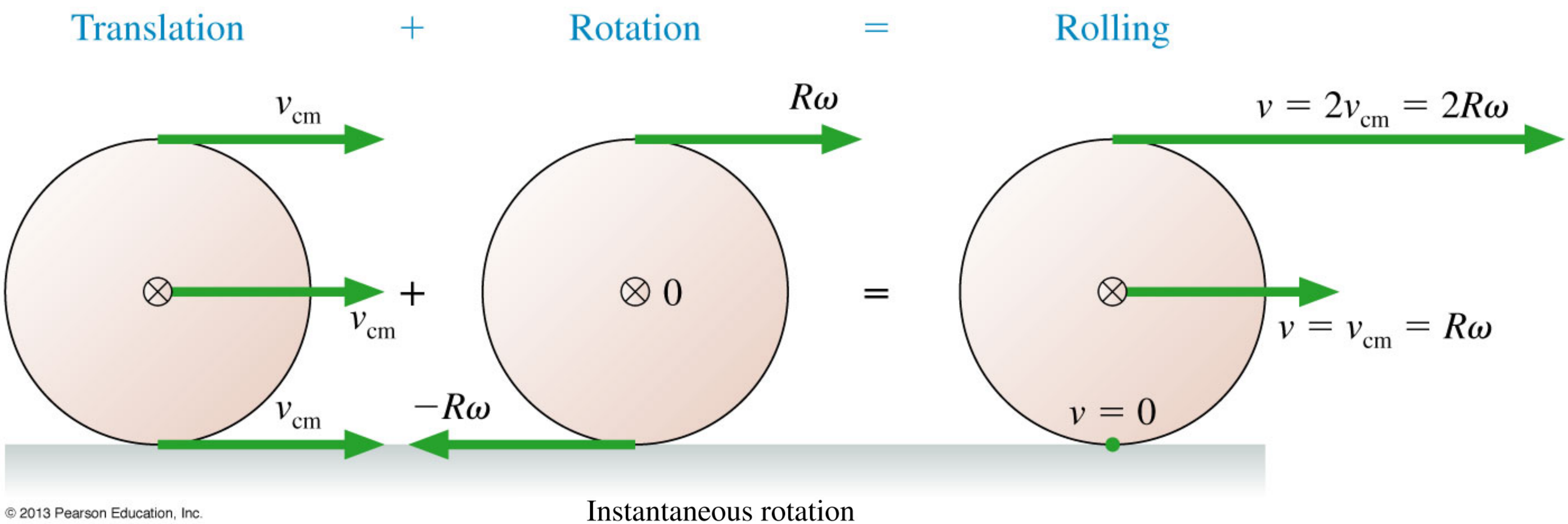


Figure 12.43


Translation
+
Rotation
=
Rolling
v_{cm}
v_{cm}
v_{cm}
+
$R\omega$
0
$-R\omega$
=
$v = 2v_{cm} = 2R\omega$
$v = v_{cm} = R\omega$
$v = 0$
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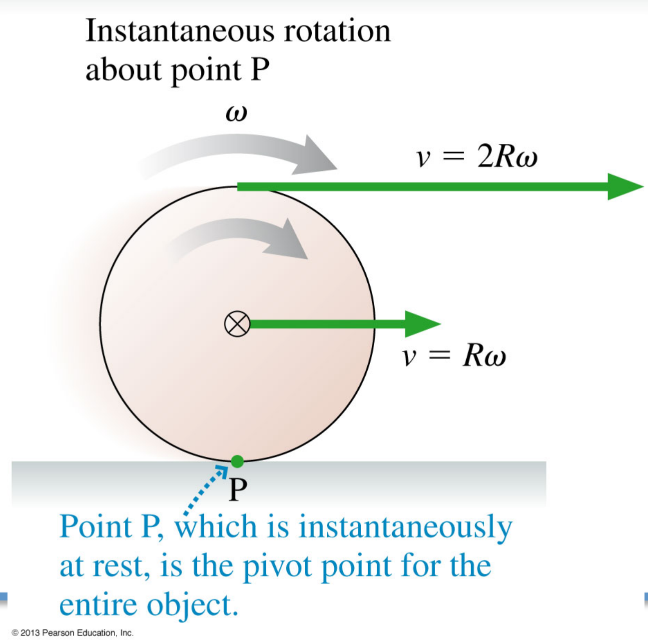
Instantaneous rotation
about point P
ω
$v = 2R\omega$
$v = R\omega$
P
Point P, which is instantaneously
at rest, is the pivot point for the
entire object.
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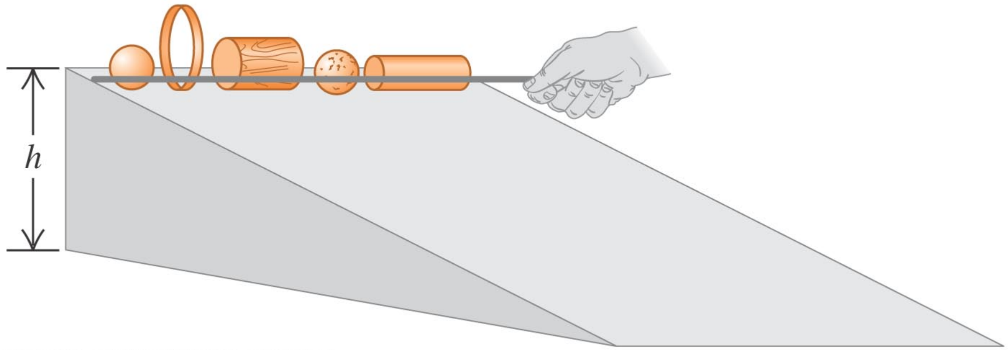
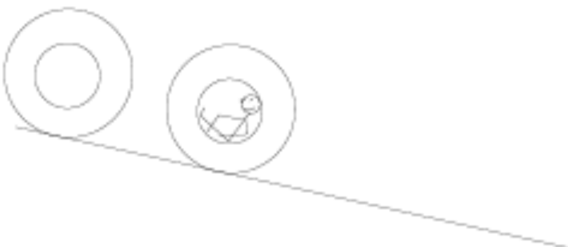
Clicker Question

- Which object rolls down the fastest?



a) Solid sphere
b) Hollow sphere
c) solid cyliner
d) ring
e) At least two object roll down with the same speed

Which will roll down a hill at a quicker rate, a large inner-tube, or an identical inner-tube with a small child squished inside the inner-tube? Assume both start from rest, and that neither inner-tube slips.



(a) The tire without the child

(b) The tire with the child

(c) Both will roll down at the same rate

Hint for HW Problem 10.70

What is minimum height for ball to not fall of track?